Distributional Itô's Formula and Regularization of Generalized Wiener Functionals

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Let $X = (X_t)_{t \geqslant 0}$ be a diffusion process defined as a solution to d-dimensional stochastic differential equation

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}w(t) + b(X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d,$$

where $w = (w^1(t), \cdots, w^d(t))_{t \geqslant 0}$ is a d-dimensional Wiener process with $w(0) = 0$. The main conditions on $\sigma = (\sigma^i_j)_{1 \leqslant i,j \leqslant d}$ and $b = (b^i)_{1 \leqslant i \leqslant d}$ under which we will work are combinations from the following.

(H1) the coefficients σ and b are C^∞, and have bounded derivatives in all orders $\geqslant 1$.

(H2) $(\sigma\sigma^*)(x)$ is strictly positive, where $x = X_0$ and σ^* is the transposed matrix of σ.

(H3) There exists $\lambda > 0$ such that

$$\lambda |\xi|^2_{\mathbb{R}^d} \leqslant \langle \xi, (\sigma\sigma^*)(x)\xi \rangle_{\mathbb{R}^d} \quad \text{for all } \xi \in \mathbb{R}^d,$$

where $\langle \bullet, \bullet \rangle_{\mathbb{R}^d}$ is the standard inner product on $\mathbb{R}^d$, and $|\bullet|_{\mathbb{R}^d} = |\bullet|$ is the corresponding norm.

(H4) There exists $\kappa > 0$ such that

$$\langle \xi, (\sigma\sigma^*)(x)\xi \rangle_{\mathbb{R}^d} \leqslant \kappa |\xi|^2_{\mathbb{R}^d} \quad \text{for all } \xi \in \mathbb{R}^d.$$

In the case of $d = 1$, many researchers in stochastic analysis would know, at least intuitively, the symbol "$\int_0^T \delta_y(X_t)\mathrm{d}t$" stands for a quantity relating to the local time of X at y (evaluated at time T) which is a random variable at each time T, even though not for $\delta_y(X_t)$. Therefore, one expects naturally that the integration with respect to time gives rise to something like a 'smoothing effect'.

One way to formulate this phenomenon might be to employ notions in Malliavin calculus. For a distribution Λ on $\mathbb{R}^d$, if $\Lambda(X_t) \in \mathbb{D}^s_p$ (where $\mathbb{D}^s_p$ stands for the Sobolev space of integrability-index p and differentiability-index $s \in \mathbb{R}$ with respect to the Malliavin derivative), we define $\int_0^T \Lambda(X_t)\mathrm{d}t$ the Bochner integral of the mapping $(0, T] \ni t \mapsto \Lambda(X_t) \in \mathbb{D}^s_p$. Here, one needs to address the Bochner integrability.

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To include local times for one-dimensional diffusions in our scope, we prepare the following

Proposition 1. *Assume* $d = 1$, (H1) *and* (H2). *Let* $\Lambda \in \mathscr{S}'(\mathbb{R})$ *be positive. Then for every* $p \in (1, \infty)$, *we have* $\int_0^T \|\Lambda(X_t)\|_{p,-2} \mathrm{d}t < +\infty$.

Hence the mapping $(0, T] \ni t \mapsto \delta_y(X_t) \in \mathbb{D}_p^{-2}$ is Bochner integrable in the case of $d = 1$. For multi-dimensional cases, it is sufficient to assume $x \neq y$ in order to guarantee the Bochner integrability.

Let $H_p^s(\mathbb{R}^d) := (1 - \triangle)^{-s/2} L_p(\mathbb{R}^d, \mathrm{d}x)$, $p \in (1, \infty)$, $s \in \mathbb{R}$ be the Bessel potential spaces. The main result in this talk is the following.

Theorem 2. *Assume* (H1), (H3) *and* (H4). *Let* $p \in (1, \infty)$ *and* $s \in \mathbb{R}$. *Then for each* $\Lambda \in H_p^s(\mathbb{R}^d)$, *we have*

(i) $\Lambda(X_t) \in \mathbb{D}_{p'}^s$ *for* $t > 0$ *and* $p' \in (1, p)$;

(ii) *if* $p > 2$, *we further have* $\int_{t_0}^T \Lambda(X_t) \mathrm{d}t \in \mathbb{D}_{p'}^{s+1}$ *for* $t_0 \in (0, T]$ *and* $p' \in [2, p)$.

Hence in this sense, we exhibited the 'smoothing effect' of the Bochner integral.

This will be shown by generalizing the classical Itô's formula to our distributional setting. For this, one needs to define the stochastic integrals where the integrand is a family of generalized Wiener functionals. We define this object as (a generalization of) Skorokhod integrals.

Now the key to prove Theorem 2 is to use the elliptic regularity theorem and to track the differentiability-index of the stochastic integrals according to that of the integrand:

Theorem 3. *Assume* (H1) *and* (H2). Let $s \in \mathbb{R}$, $p \geqslant 2$ and $\Lambda \in H_p^s(\mathbb{R}^d)$. Then we have

$$\int_0^T \Lambda(X_t) \mathrm{d}w^i(t) \in \mathbb{D}_p^s, \quad \text{for } i = 1, \cdots, d$$

provided either one of the following

(i) $\lim_{t \downarrow 0} \|\Lambda(X_t)\|_{p,s} = 0$.

(ii) $s \geqslant 0$ and $\int_0^T \|\Lambda(X_t)\|_{p,s}^2 \mathrm{d}t < \infty$.

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