Smooth approximation of a Yang–Mills theory on $\mathbb{R}^2$: a rough path approach

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Let C be the set of smooth curves $\mathsf{c} : \mathbb{R} \to \mathbb{R}^2$. Let $G = SU(n_{\mathrm{mat}})$ ($n_{\mathrm{mat}} \geq 2$), and $\mathfrak{g} = \mathfrak{su}(n_{\mathrm{mat}})$), the Lie algebra of G equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, minus the Killing form. Let $\Omega^1 = \Omega^1(\mathbb{R}^2, \mathfrak{g})$ denote the space of $\mathfrak{g}$-valued smooth 1-forms on $\mathbb{R}^2$. Let $A = A_1 dx^1 + A_1 dx^2 \in \Omega^1$ ($A_1, A_2 \in C^\infty(\mathbb{R}^2, \mathfrak{g})$). The *parallel transport* $h_{\mathsf{c},A}(t) \in G$ ($t \in \mathbb{R}$) along $\mathsf{c} \in \mathsf{C}$ is defined by the differential equation

$$\frac{dh_{\mathsf{c},A}(t)}{dt} = A\left(\dot{\mathsf{c}}(t)\right) h_{\mathsf{c},A}(t) = \sum_{k=1}^{2} A_k(\mathsf{c}(t))\dot{\mathsf{c}}_k(t) h_{\mathsf{c},A}(t), \qquad h_{\mathsf{c},A}(0) = 1_G \tag{1}$$

Conjecture 1. *There exists a sequence of Ω^1-valued random variables $A^{(j)}$ ($j \in \mathbb{N}$) on a probability space $(\mathbb{P}, \Omega)$, and a complete metric space $(\mathcal{G}, d)$ with $\mathcal{G} \subset C(\mathbb{R}, G)$ such that*

(1) $\mathbb{P}\left[h_{\mathsf{c}} := \lim_{j\to\infty} h_{\mathsf{c},A^{(j)}}\right.$ *exists in* $\mathcal{G}$ *for all* $\left.\mathsf{c} \in \mathsf{C}\right] = 1$.

(2) The set of G-valued random variables $\{h_{\mathsf{c}} : \mathsf{c} \in \mathsf{C},\ \mathsf{c}$ is a loop$\}$ obeys the law of the Wilson loops of Yang–Mills (YM) theory on $\mathbb{R}^2$ (see e.g. [1, 5, 6, 4]).

Generally a YM theory is formulated on a Riemannian manifold (mainly with dimension ≤ 4). YM on $\mathbb{R}^2$ is the simplest (and physically trivial) case of the YM theory; nevertheless, the rigorous proof of the above conjecture does not seem easy. We will give a partial result on this conjecture.

Let $\boldsymbol{\Delta}_i$ ($i \geq -1$) be the Littlewood–Paley block, and $\mathcal{M}_j := \sum_{i \leq j-1} \boldsymbol{\Delta}_i$ be the jth 'mollification' operator on $\mathscr{S}'(\mathbb{R}^2)$, which satisfies $\lim_{j\to\infty} \mathcal{M}_j u = u$.

Let W be a $\mathfrak{g}$-valued standard Gaussian white noise on $\mathbb{R}^2$. Define the jth *smooth approximation* $W^{(j)} \in C^\infty(\mathbb{R}^2, \mathfrak{g})$ of W by $W^{(j)} := \mathcal{M}_j W$. Define the Ω^1-valued random variable $A^{(j)} = A_1^{(j)} dx_1 + A_2^{(j)} dx_2 \in \Omega^1$ ($A_1^{(j)}, A_2^{(j)} \in C^\infty(\mathbb{R}^2, \mathfrak{g})$) by

$$A_1^{(j)}(x) \equiv 0, \quad A_2^{(j)}(x) := \int_0^{x^1} W^{(j)}(\xi, x_2) d\xi, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

The condition $A_1^{(j)}(x) \equiv 0$ is called the *axial gauge condition.*

Let $C^{p\text{-var}}([s,t], G^2(\mathbb{R}^d))$ ($p \in (2,3)$) denote the space of p-variation weak geometric rough paths. For $x \in C^{1\text{-var}}([s,t], \mathbb{R}^d)$ (the space of continuous functions of bounded variation), let $S_2(x) \in C^{p\text{-var}}([s,t], G^2(\mathbb{R}^d))$ be the step-2 canonical lift of x (see [3, 2]).

Definition 2. Let $V : \mathbb{R}^e \to L(\mathbb{R}^d, \mathbb{R}^e)$. Let $(x^{(n)})_n$ be a sequence in $C^\infty([0,T], \mathbb{R}^d)$. $y \in C([0,T], \mathbb{R}^e)$ is called a *FV solution* of the (formal) ODE

$$dy = V(y) dx^{(\cdot)}, \quad y(0) = y_0 \in \mathbb{R}^e \tag{2}$$

if the limit

$$\lim_{n\to\infty} S_2(x^{(n)}) =: \mathbf{x} \in C^{p\text{-var}}([0,T], G^2(\mathbb{R}^d))$$

exists, and y is a solution of the rough differential equation (RDE)

$$dy = V(y) d\mathbf{x}, \quad y(0) = y_0 \in \mathbb{R}^e$$

in the Friz–Victoir (FV) sense [3, Def. 10.17]. If the solution is unique we write $y = \Pi_{(V)}((x^{(n)}), y_0)$.

We see Eq. (1) is rewritten as

$$dh_{\mathsf{c},A} = V(h_{\mathsf{c},A})dX, \quad h_{\mathsf{c},A}(0) = 1_G \in G, \quad X(t) = X_{\mathsf{c},A}(t) := \int_{\mathsf{c}\restriction[0,t]} A.$$

where $V(M)$ ($M \in \mathrm{Mat}(n_{\mathrm{mat}}, \mathbb{C})$) is the linear operator on $\mathrm{Mat}(n_{\mathrm{mat}}, \mathbb{C}) \cong \mathbb{R}^{2n_{\mathrm{mat}}^2}$ defined by $V(M)N := NM$, $N \in \mathrm{Mat}(n_{\mathrm{mat}}, \mathbb{C})$.

Let $\mathsf{C}_{\mathrm{nice}} \subset \mathsf{C}$ be a set of 'well-behaved' curves in C (roughly speaking, the curve $\mathsf{c} \in \mathsf{C}_{\mathrm{nice}}$ does not rotate around a point in $\mathbb{R}^2$ infinitely many times). Let $X_{\mathsf{c}}^{(j)} := X_{\mathsf{c},A^{(j)}}$ and $\mathbf{X}_{\mathsf{c}}^{(j)} = (1, X_{\mathsf{c}}^{(j)}, \mathbb{X}_{\mathsf{c}}^{(j)}) := S_2(X_{\mathsf{c}}^{(j)})$.

Lemma 3 (rough path convergence in L^p)**.** *Let $\alpha = 1/p \in (1/3, 1/2)$ and $q \in [1, \infty)$. Suppose $\mathsf{c} \in \mathsf{C}_{\mathrm{nice}}$. Then there exists $\mathbf{X}_{\mathsf{c}} \in C^{\alpha\text{-Höl}}([0,T], G^2(\mathfrak{g}))$ such that $\mathbf{X}_{\mathsf{c}}^{(j)} \to \mathbf{X}_{\mathsf{c}}$ in $C^{\alpha\text{-Höl}}([0,T], G^2(\mathfrak{g}))$ and $L^q(\mathbb{P})$, i.e.*

$$\lim_{j\to\infty} \left\| d_{\mathrm{CC},\alpha\text{-Höl};[0,T]}\big(\mathbf{X}_{\mathsf{c}}, \mathbf{X}_{\mathsf{c}}^{(j)}\big) \right\|_{L^q(\mathbb{P})} = 0,$$

where $d_{\mathrm{CC},\alpha\text{-Höl};[0,T]}$ denotes the canonical metric on $C^{\alpha\text{-Höl}}([0,T], G^2(\mathfrak{g}))$ (α-Hölder Carnot–Carathéodory metric).

Theorem 4. *There exists a subsequence $\big(A^{(j_n)}\big)_{n\in\mathbb{N}}$ of $\big(A^{(j)}\big)_j$ such that for any finite subset $S \subset \mathsf{C}_{\mathrm{nice}}$*

(i) $\mathbb{P}\Big[h_{\mathsf{c}} := \Pi_{(V)}((X_{\mathsf{c}}^{(j_n)})_n, 1_G) \text{ exists for all } \mathsf{c} \in S\Big] = 1,$

(ii) The set of G-valued random variables $\{h_{\mathsf{c}} : \mathsf{c} \in S,\ \mathsf{c} \text{ is a loop}\}$ obeys the law of the Wilson loops of YM theory on $\mathbb{R}^2$.

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