Malliavin calculus for conditional intensities of Hawkes processes

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1. Preliminaries

$\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$

- $\{\tau_k\,;\,k \geq 1\}$: the simple point process

  (i.e. τ_k: the $(0,+\infty)$-valued random variable, $\tau_k < \tau_{k+1}$ a.s.)

- $\{N_t\,;\,t \geq 0\}$: the counting process defined by

$$N_t = \sum_{k\geq 1} \mathbb{I}_{(0,t]}(\tau_k) \tag{1}$$

- $\{Z_k\,;\,k \geq 1\}$: the sequence of independent and identically distributed $\mathbb{R}_0$-valued random variables

Assumption 1

(i) the law of Z_1 has a C^∞-density $f(z)$ such that $\lim_{|z|\to+\infty} f(z) = 0$,

(ii) the sequences $\{\tau_k\,;\,k \geq 1\}$ and $\{Z_k\,;\,k \geq 1\}$ are independent.

- $\{(\tau_k,\ Z_k)\,;\ k \geq 1\}$: the marked point process
- $\{L\big((0,t], A\big)\,;\ t \geq 0,\ A \in \mathcal{B}(\mathbb{R}_0)\big\}$: the counting process defined by

$$L((0,t], A) = \sum_{k\geq 1} \mathbb{I}_{(0,t]}(\tau_k)\,\mathbb{I}_A(Z_k) \left(= \sum_{k=1}^{N_t} \mathbb{I}_A(Z_k) \right) \qquad (2)$$

- $\{\lambda(t, A)\,;\ t \geq 0,\ \ A \in \mathcal{B}(\mathbb{R}_0)\}$: the conditional intensity given by

$$\lambda(t, A) = \lim_{\delta \searrow 0} \mathbb{E}\left[\frac{L\big((0,t+\delta], A\big) - L\big((0,t], A\big)}{\delta} \,\middle|\, \mathcal{F}_t \right] \qquad (3)$$

$$\left(\mathcal{F}_t := \bigcap_{\varepsilon > 0} \sigma\big[L\big((0,s], A\big)\,;\ s \leq t+\varepsilon,\ A \in \mathcal{B}(\mathbb{R}_0)\big] \vee \mathcal{N} \right)$$

- $\{\Lambda\big((0,t], A\big)\,;\ t \geq 0,\ \ A \in \mathcal{B}(\mathbb{R}_0)\}$: the compensator defined by

$$\Lambda\big((0,t], A\big) := \int_0^t \lambda(s, A)\,\mathrm{d}s \qquad (4)$$

Remark 1

(1) *For each* $t \geq 0$,

$$L\big((0,t],\mathbb{R}_0\big) = \sum_{k\geq 1} \mathbb{I}_{(0,t]}(\tau_k)\,\mathbb{I}_{\mathbb{R}_0}(Z_k) = N_t.$$

(2) *for each* $A \in \mathcal{B}(\mathbb{R}_0)$, *the process*

$$\Big\{\tilde{L}\big((0,t],A\big) := L\big((0,t],A\big) - \Lambda\big((0,t],A\big)\,;\, t \geq 0\Big\}$$

is $\{\mathcal{F}_t\,;\, t \geq 0\}$*-martingale.* □

$$L\big((0,t],A\big) = \int_0^t \int_A L(\mathrm{d}s,\mathrm{d}z) \qquad \lambda(t,A) = \int_A \lambda(t,\mathrm{d}z)$$

$$\Lambda\big((0,t],A\big) = \int_0^t \int_A \Lambda(\mathrm{d}s,\mathrm{d}z) = \int_0^t \int_A \lambda(s,\mathrm{d}z)\,\mathrm{d}s$$

$$\tilde{L}\big((0,t],A\big) = \int_0^t \int_A \tilde{L}(\mathrm{d}s,\mathrm{d}z)$$

2. Hawkes processes

$\pi(\mathrm{d}z)$: the finite measure on $\mathbb{R}_0$

$h : [0, +\infty) \to [0, +\infty)$: bounded and Borel measurable

Assumption 2

$$\|h\|_1 \left(:= \int_0^{+\infty} h(s)\,\mathrm{d}s \right) < 1.$$

Hawkes process

$\{L((0,t], A)\,;\, t \geq 0,\ A \in \mathcal{B}(\mathbb{R}_0)\}$ is called *the marked Hawkes process* with the conditional intensity $\{\lambda(t, A)\,;\, t \geq 0,\ A \in \mathcal{B}(\mathbb{R}_0)\}$, if

$$\lambda(t, A) = \pi(A) + \int_0^t \int_A h(t-s)\, L(\mathrm{d}s, \mathrm{d}z) \tag{5}$$

$$\left(\iff \lambda(t, A) = \pi(A) + \sum_{k \geq 1} h(t - \tau_k)\, \mathbb{I}_{(0,t]}(\tau_k)\, \mathbb{I}_A(Z_k) \right)$$

$\ell \in C_b^\infty(\mathbb{R}_0\,;\,[0,+\infty))$

- $\{L_t\,;\,t \geq 0\}$, $L_t = \int_0^t \int_{\mathbb{R}_0} \ell(z)\, L(\mathrm{d}s, \mathrm{d}z) \left(\equiv \sum_{k=1}^{N_t} \ell(Z_k) \right)$
- $\{\lambda_t\,;\,t \geq 0\}$, $\lambda_t = \int_{\mathbb{R}_0} \ell(z)\, \lambda(t, \mathrm{d}z)$
- $\{\Lambda_t\,;\,t \geq 0\}$, $\Lambda_t = \int_0^t \int_{\mathbb{R}_0} \ell(z)\, \Lambda(\mathrm{d}s, \mathrm{d}z) \left(= \int_0^t \int_{\mathbb{R}_0} \ell(z)\, \lambda(s, \mathrm{d}z)\, \mathrm{d}s \right)$

Write $\mu = \int_{\mathbb{R}_0} \ell(z)\, \pi(\mathrm{d}z)$. Then, from (5), we see that

$$\lambda_t = \mu + \int_0^t \int_{\mathbb{R}_0} h(t-s)\, \ell(z)\, L(\mathrm{d}s, \mathrm{d}z) \tag{6}$$

$$\left(= \mu + \sum_{k=1}^{N_t} h(t-\tau_k)\, \ell(Z_k) \right)$$

$\Psi(u) = \sum_{k\geq 1} h^{*k}(u)$

Proposition 2

$$\lambda_t = \mu + \int_0^t \Psi(t-s)\,\mu\,ds + \int_0^t \Psi(t-s)\,d\tilde{L}_s \tag{7}$$

Proof of Proposition 2. Write $\gamma_t = \mu + \int_0^t h(t-s)\,\mathrm{d}\tilde{L}_s$. Since

$$\int_0^t h(t-s)\left(\int_0^s h(s-u)\,\lambda_u \mathrm{d}u\right)\mathrm{d}s = \int_0^t h^{*2}(t-s)\,\lambda_s\,\mathrm{d}s$$

from the Fubini theorem, we see that

$$\begin{aligned}\lambda_t &\left(= \mu + \int_0^t\int_{\mathbb{R}_0} h(t-s)\,\ell(z)\,L(\mathrm{d}s,\mathrm{d}z)\right) = \mu + \int_0^t h(t-s)\,\mathrm{d}L_s \\ &= \gamma_t + \int_0^t h(t-s)\,\lambda_s\,\mathrm{d}s \\ &= \gamma_t + \int_0^t h(t-s)\,\gamma_s\,\mathrm{d}s + \int_0^t h^{*2}(t-s)\,\lambda_s\,\mathrm{d}s = \cdots \\ &= \gamma_t + \int_0^t \Psi(t-s)\,\gamma_s\,\mathrm{d}s = \text{(the right hand side)}.\end{aligned}$$

□

Corollary 1

$$\mathbb{E}[\lambda_t] = \mu + \mu \int_0^t \Psi(s)\, ds \tag{8}$$

$$\lim_{t \to +\infty} \mathbb{E}[\lambda_t] = \frac{\mu}{1 - \|h\|_1} \tag{9}$$

Proof of Colollary 1. The first assertion can be justified by Proposition 2. Assumption 2 enables us to get the second assertion, because

$$\|\Psi\|_1 = \sum_{k \geq 1} \left\|h^{*k}\right\|_1 = \frac{\|h\|_1}{1 - \|h\|_1}.$$

□

Proposition 3

For each $t \geq 0$, the random variable λ_t is in $\bigcap_{p \geq 1} \mathbb{L}^p(\Omega)$.

Proof of Proposition 3. The assertion can be proved by the boundedness of the functions h and ℓ. □

3. Integration by parts formula

$$\lambda_t = \mu + \sum_{k=1}^{N_t} h(t - \tau_k)\, \ell(Z_k)$$

Goal

the integration by parts formula for λ_t

[the Malliavin calculus for jump processes]

- the Bismut approach based upon the Girsanov transform
- the Picard approach by using the difference operator
- the calculus focused on the jump-size components

Assumption 4

1. $f(0+) = f(0-)$, $f'/f \in C_b^\infty(\mathbb{R}_0)$,
2. $\ell'(0+) = \ell'(0-)$,
3. there exist a constant $C_1 > 0$ such that $\inf_{z \in \mathbb{R}_0} |\ell'(z)| \geq C_1$,
4. there exists $C_2 > 0$ such that $\inf_{t \in [0,T]} h(t) \geq C_2$.

$$\lambda_t^{(N,j)} = \mu + \sum_{j \neq k = 1}^{N} h(t - \tau_k)\, \ell(Z_k) + h(t - \tau_j)\, \ell(z_j),$$

$$\Gamma_t^{(N,j)} = \sum_{j \neq k = 1}^{N} h(t - \tau_k)^2\, \ell'(Z_k)^2 + h(t - \tau_j)^2\, \ell'(z_j)^2,$$

$$\lambda_t^{(N)} = \lambda_t^{(N,j)}\big|_{z_j = Z_j}, \quad \Gamma_t^{(N)} = \Gamma_t^{(N,j)}\big|_{z_j = Z_j}.$$

Theorem 1 (Integration by parts formula)

Let $\phi \in C_b^1(\mathbb{R})$ *and* $0 < t \leq T$. *Then, it holds that*

$$\mathbb{E}\big[\phi'(\lambda_t)\,\mathbb{I}_{(N_t \geq 1)}\big] = \mathbb{E}\big[\phi(\lambda_t)\,\Theta_t\,\mathbb{I}_{(N_t \geq 1)}\big]. \tag{10}$$

$$\left(\Theta_t = -\sum_{j=1}^{N_t} \frac{1}{f(Z_j)}\,\partial_j \left(\frac{h(t-\tau_j)\,\ell'(z_j)\,f(z_j)}{\Gamma_t^{(N_t,j)}} \right)\Bigg|_{z_j = Z_j}\right)$$

Remark 4

Under $\{N_t \geq 1\}$, *it holds that*

$$\begin{aligned}\Gamma_t^{(N_t,j)} &= \sum_{j \neq k = 1}^{N_t} h(t-\tau_k)^2\,\ell'(Z_k)^2 + h(t-\tau_j)^2\,\ell'(z_j)^2 \\ &\geq C_1^2\,C_2^2. \qquad (\textit{by Assumption 4})\end{aligned}$$

□

Proof of Theorem 1. Let $\phi \in C_b^1(\mathbb{R})$ and $0 < t \le T$. Then, we have

$$\mathbb{E}\big[\phi'(\lambda_t)\,\mathbb{I}_{(N_t \ge 1)}\big] = \sum_{N=1}^{+\infty} \mathbb{E}\Big[\phi'\big(\lambda_t^{(N)}\big)\,\mathbb{I}_{(N_t = N)}\Big].$$

The integration by parts tells us to see that

$$\mathbb{E}\Big[\phi'\big(\lambda_t^{(N)}\big)\,\mathbb{I}_{(N_t=N)}\Big] = \sum_{j=1}^{N} \mathbb{E}\left[\phi'\big(\lambda_t^{(N)}\big)\,\frac{h(t-\tau_j)^2\,\ell'(Z_j)^2}{\Gamma_t^{(N)}}\,\mathbb{I}_{(N_t=N)}\right]$$

$$= \sum_{j=1}^{N} \mathbb{E}\left[\partial_j\Big(\phi\big(\lambda_t^{(N,j)}\big)\Big)\bigg|_{z_j=Z_j}\,\frac{h(t-\tau_j)\,\ell'(Z_j)}{\Gamma_t^{(N)}}\,\mathbb{I}_{(N_t=N)}\right]$$

$$= \sum_{j=1}^{N} \mathbb{E}\left[\int_{\mathbb{R}_0} \partial_j\Big(\phi\big(\lambda_t^{(N,j)}\big)\Big)\,\frac{h(t-\tau_j)\,\ell'(z_j)}{\Gamma_t^{(N,j)}}\,f(z_j)\,\mathrm{d}z_j\,\mathbb{I}_{(N_t=N)}\right]$$

$$= -\sum_{j=1}^{N} \mathbb{E}\left[\int_{\mathbb{R}_0} \phi\big(\lambda_t^{(N,j)}\big)\,\partial_j\left(\frac{h(t-\tau_j)\,\ell'(z_j)}{\Gamma_t^{(N,j)}}\,f(z_j)\right)\,\mathrm{d}z_j\,\mathbb{I}_{(N_t=N)}\right]$$

$$= -\mathbb{E}\left[\phi\big(\lambda_t^{(N)}\big)\,\sum_{j=1}^{N}\frac{1}{f(Z_j)}\,\partial_j\left(\frac{h(t-\tau_j)\,\ell'(z_j)}{\Gamma_t^{(N,j)}}\,f(z_j)\right)\bigg|_{z_j=Z_j}\mathbb{I}_{(N_t=N)}\right].$$

□

Theorem 2

Let $m \in \mathbb{N}$ *and* $\phi \in C_b^m(\mathbb{R})$ *and* $0 < t \leq T$. *Then, it holds that*

$$\mathbb{E}\big[\phi^{(m)}(\lambda_t)\,\mathbb{I}_{(N_t \geq 1)}\big] = \mathbb{E}\big[\phi(\lambda_t)\,\Theta_t^{(m)}\,\mathbb{I}_{(N_t \geq 1)}\big], \tag{11}$$

where

$$\Theta_t^{(m)} = \begin{cases} \Theta_t & (m = 1), \\ -\displaystyle\sum_{j=1}^{N_t} \frac{1}{f(Z_j)}\,\partial_j \left(\frac{\Theta_t^{(m-1,j)}\,h(t-\tau_j)\,\ell'(z_j)\,f(z_j)}{\Gamma_t^{(N_t,j)}} \right)\Bigg|_{z_j = Z_j} & (m \geq 2). \end{cases}$$

Proof of Theorem 2. Iterative application of Theorem 1:

$$\begin{aligned} (\text{the left hand side}) &= \mathbb{E}\big[\phi^{(m-1)}(\lambda_t)\,\Theta_t^{(1)}\,\mathbb{I}_{(N_t \geq 1)}\big] \\ &= \mathbb{E}\big[\phi^{(m-2)}(\lambda_t)\,\Theta_t^{(2)}\,\mathbb{I}_{(N_t \geq 1)}\big] \\ &= \cdots \\ &= (\text{the right hand side}). \end{aligned}$$

□

Theorem 3

Under Assumptions 1 and 4, for each $0 < t \leq T$, the conditional law of λ_t admits a C^∞-Lebesgue density under $\{N_t \geq 1\}$.

Proof of Theorem 3. Direct consequence of Theorems 1 and 2. Remark that the inverse of $\Theta_t^{(m)}$ can be checked by the assumptions. □

Theorem 4

$$\mathbb{P}\big[\lambda_t \in dy \big| N_t \geq 1\big] = \frac{\mathbb{E}\big[\mathbb{I}_{[y,+\infty)}(\lambda_t)\,\Theta_t\,\mathbb{I}_{(N_t \geq 1)}\big]}{\mathbb{P}\big[N_t \geq 1\big]}\,dy. \tag{12}$$

Proof of Theorem 4. Let $g \in C_0^\infty(\mathbb{R})$ be non-negative, and write $G(x) = \int_{-\infty}^x g(y)\,\mathrm{d}y$. From Theorem 1, we have

$$\begin{aligned}
\mathbb{E}\big[g(\lambda_t)\,\mathbb{I}_{(N_t\geq 1)}\big] &= \mathbb{E}\big[G'(\lambda_t)\,\mathbb{I}_{(N_t\geq 1)}\big] = \mathbb{E}\big[G(\lambda_t)\,\Theta_t\,\mathbb{I}_{(N_t\geq 1)}\big] \\
&= \int_{\mathbb{R}} g(y)\,\mathbb{E}\big[\mathbb{I}_{[y,+\infty)}(\lambda_t)\,\Theta_t\,\mathbb{I}_{(N_t\geq 1)}\big]\,\mathrm{d}y.
\end{aligned}$$

□

5. Error estimate

$T > 0$, $\xi \geq 0$: fixed

$$L_t = \sum_{k=1}^{N_t} \ell(Z_k), \quad \lambda_t = \int_{\mathbb{R}_0} \ell(z)\,\pi(\mathrm{d}z) + \sum_{k=1}^{N_t} h(t - \tau_k)\,\ell(Z_k).$$

Assumption 5

$$\pi(\mathrm{d}z) = f(z)\,\mathrm{d}z, \qquad \|h\|_{\infty,T} \left(:= \sup_{0 \leq t \leq T} |h(t)| \right) < 1.$$

$\{\overline{L}_t\,;\, t \in [0,T]\}$: the compound Poisson process such that

$$\mathbb{E}\left[e^{-\xi\,\overline{L}_t}\right] = \exp\left(t\,\tilde{\mu}_\xi\right), \tag{13}$$

where $\ell_\xi(z) = e^{-\xi\,\ell(z)} - 1$ and $\tilde{\mu}_\xi = \int_{\mathbb{R}_0} \ell_\xi(z)\,\pi(\mathrm{d}z)$.

Proposition 5

$$\left|\mathbb{E}\left[e^{-\xi L_t}\right] - \mathbb{E}\left[e^{-\xi \overline{L}_t}\right]\right| \leq \frac{\xi\,\mu\,T^2\,\|h\|_{T,\infty}}{2\,(1-\|h\|_1)}\,e^{t\tilde{\mu}_\xi}. \tag{14}$$

Proof of Proposition 5. Remark that

$$\mathbb{E}\left[e^{-\xi \overline{L}_t}\right] = 1 + \tilde{\mu}_\xi \int_0^t \mathbb{E}\left[e^{-\xi \overline{L}_s}\right] \mathrm{d}s.$$

Since $e^{-\xi L_t} = 1 + \int_0^t \int_{\mathbb{R}_0} e^{-\xi L_{s-}}\,\ell_\xi(z)\,L(\mathrm{d}s, \mathrm{d}z)$, we have

$$\begin{aligned}
\mathbb{E}\left[e^{-\xi L_t}\right] &= 1 + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} e^{-\xi L_{s-}}\,\ell_\xi(s)\,\lambda(s, \mathrm{d}z)\,\mathrm{d}s\right] \\
&= 1 + \tilde{\mu}_\xi \int_0^t \mathbb{E}\left[e^{-\xi L_s}\right] \mathrm{d}s \\
&\quad + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} e^{-\xi L_{s-}}\,\ell_\xi(z)\left(\int_0^s h(s-u)\,L(\mathrm{d}u, \mathrm{d}z)\right) \mathrm{d}s\right].
\end{aligned}$$

□

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