On the Euler-Maruyama scheme for SDEs with discontinuous diffusion coefficient

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joint work with
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Outline

Introduction

- Let $X = (X_t)_{0 \le t \le T}$ be a solution of the one-dimensional SDE

$$X_t = x_0 + \int_0^t \sigma(X_s) dW_s, \; x_0 \in \mathbb{R}, \; t \in [0, T], \tag{1}$$

- $W := (W_t)_{0 \le t \le T}$: standard one-dimensional Brownian motion
- diffusion coefficient $\sigma : \mathbb{R} \to \mathbb{R}$.

Definition 1

The Euler-Maruyama approximation $X^{(n)} = (X_t^{(n)})_{0 \le t \le T}$ of equation (1) is defined by

$$X_t^{(n)} = x_0 + \int_0^t \sigma(X_{\eta(s)}^{(n)}) dW_s$$
$$= X_{\eta_n(t)}^{(n)} + \sigma(X_{\eta_n(t)}^{(n)})(W_t - W_{\eta_n(t)}),$$

where $\eta(s) = kT/n$ if $s \in [kT/n, (k+1)T/n)$.

- Note that $X_0^{(n)} = x_0$, and for any $k = 1, \dots, n$,

$$X_{kT/n}^{(n)} = X_{(k-1)T/n}^{(n)} + \sigma(X_{(k-1)T/n}^{(n)})(W_{kT/n} - W_{(k-1)T/n})$$

and

$$X_{(k-1)T/n}^{(n)} \text{ and } \underbrace{(W_{kT/n} - W_{(k-1)T/n})}_{\sim N(0, T/n)} \text{ are independent.}$$

$\Rightarrow$ We can simulate the random variable $X_T^{(n)}$.

Maruyama[1] introduce the approximation in order to prove Girsanov's theorem (Cameron-Martin-Maruyama-Girsanov theorem) for the solution of one-dimensional SDE $\mathbf{d}X_t = b(X_t)\mathbf{d}t + \mathbf{d}W_t$.

[1]On the transition probability functions of the Markov process., Nat. Sci. Rep. Ochanomizu Univ. 5, 10-20. (1954).

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Theorem 1 (Kanagawa (1988), Faure (1992), Kloeden and Platen (1992))

If the coefficient σ is Lipschitz continuous then the Euler-Maruyama approximation has a strong rate of order $1/2$, i.e., for any $p \geq 1$,

$$\mathbb{E}[\sup_{0\leq t\leq T} |X_t - X_t^{(n)}|^p]^{1/p} \leq \frac{C_p}{n^{1/2}}.$$

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Theorem 2 (Kaneko and Nakao 1988[2])

$d \geq 1$. Suppose the coefficient σ is continuous and linear growth. Under the pathwise uniqueness for the solution of SDE, it holds that

$$\lim_{n\to\infty} \mathbb{E}[\sup_{0\leq t\leq T} |X_t - X_t^{(n)}|^2] = 0.$$

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Theorem 3 (Yamada and Watanabe 1971[3])

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[5] Strong rate of convergence for the Euler-Maruyama approximation of stochastic differential equations with irregular coefficients. Math. Comp. 85(300), 1793–1819 (2016).

[6] On the Euler-Maruyama approximation for one-dimensional stochastic differential equations with irregular coefficients. To appear in IMA Journal of Numerical Analysis.

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Theorem 4 (Gyöngy and Rásonyi, 2011[4])

Suppose that the diffusion σ is α-Hölder continuous wiht $\alpha \in [1/2, 1]$. Then there exists a constant C such that

$$\sup_{0 \le t \le T} \mathbb{E}[|X_t - X_t^{(n)}|] \le \begin{cases} \dfrac{C}{n^{\alpha - 1/2}} & \text{if } \alpha \in (1/2, 1], \\ \dfrac{C}{\log n} & \text{if } \alpha = 1/2. \end{cases}$$

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• Ngo and Taguchi prove the statements in Thm 4 hold for SDEs with discont. drift, σ:UE [5] [6]

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Non-pahtwise uniquneness, Non-strong solution, Weak existence

Example 2 (Girsanov)

Let $\alpha \in (0, 1/2)$. For the SDE $dX_t = |X_t|^\alpha dW_t$ with $X_0 = 0$, the pathwise uniqueness does not hold.

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Example 3 (Tanaka's equation)

Let X be a Brownian motion. Define $W_t := \int_0^t \mathrm{sgn}(X_s) dX_s$ (BM). Then, $X_t = \int_0^t \mathrm{sgn}(X_s) dW_s$ but X does not admit a strong solution. (If X is strong sol, then $\mathcal{F}_t^X \subset \mathcal{F}_t^{|X|}$.)

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Theorem 5 (Engelbert and Schmidt 1984)

Define

$$I(\sigma) := \left\{ x \in \mathbb{R}; \forall \varepsilon > 0, \int_{-\varepsilon}^{\varepsilon} \frac{dy}{\sigma^2(x+y)} = \infty \right\}, \quad Z(\sigma) := \{x \in \mathbb{R}; \sigma(x) = 0\}.$$

The SDE (1) *($dX_t = \sigma(X_t) dW_t$) has a non-exploding weak sol. which is unique in the sense of probability law if and only if $I(\sigma) = Z(\sigma)$.*

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Remark 1

If $0 < \underline{\sigma} \le \sigma(x) \le \overline{\sigma}$, then $I(\sigma) = Z(\sigma) = \emptyset$.

Pathwise uniqueness

Assumption 1

(i) *σ is measurable, bounded and uniformly positive, i.e. there exist $\overline{\sigma}, \underline{\sigma} > 0$ such that for any $x \in \mathbb{R}$,*

$$\underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}.$$

(ii) *[bounded 2-variation] There exists a bounded and strictly increasing function f_σ such that for any $x, y \in \mathbb{R}$,*

$$|\sigma(x) - \sigma(y)| \leq |f_\sigma(x) - f_\sigma(y)|^{1/2}.$$

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[7] One-dimensional stochastic differential equations involving the local times of the unknown process. In Stochastic analysis and applications (pp. 51-82). Springer Berlin Heidelberg.

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Theorem 6 (Le Gall 1984[7])

Under Assumption 1, the pathwise uniqueness holds for SDE (1).

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Problem:
Under Assumption 1,

(Q1)

$$\lim_{n\to\infty} \mathbb{E}[|X_t - X_t^{(n)}|] = 0 \ ?$$

(Q2)

$$\mathbb{E}[|X_t - X_t^{(n)}|] \leq \frac{C}{\log n} \ ?$$

Main result

Assumption 2

Suppose that

$$\sigma = \rho \circ f,$$

where ρ *is* $1/2$*-Hölder continuous with* $0 < \underline{\sigma} \leq \rho(x) \leq \overline{\sigma}$ *and* $f = f_1 - f_2$, f_i*: bdd, strictly increasing with finite dis-conti. points*.

[8] On maps of bounded p-variation with $p > 1$. Positivity, 1998, Volume 2, Issue 1, 19-45.

[9] Strong convergence for the Euler-Maruyama approximation of stochastic differential equations with discontinuous coefficients. Preprint, arXiv:1604.01174v2

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- Ass. 2 $\Rightarrow$ Ass. 1 with $f_\sigma = ||\rho||_{1/2}^2\{f_1 + f_2\}$. Indeed,

$$|\sigma(x) - \sigma(y)| \leq ||\rho||_{1/2}\{|f_1(x) - f_1(y)| + |f_2(x) - f_2(y)|\}^{1/2} = |f_\sigma(x) - f_\sigma(y)|^{1/2}.$$

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- Structural Theorem: Chistyakov and Galkin[8] prove that $g : E \to X$ is of bounded p-variation if and only if $g = \rho \circ f$, where ρ is $1/p$-Hölder conti. f is nondecreasing, E is nonempty subset of $\mathbb{R}$ and X is metric space.

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Theorem 7 (Ngo and Taguchi, 2016, preprint[9])

Suppose Ass. 2. Then there exists $C > 0$ *such that*

$$\sup_{0 \leq t \leq T} \mathbb{E}[|X_t - X_t^{(n)}|] \leq \frac{C}{\log n}, \ \forall n \geq 3.$$

---

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Idea of proof

Standard proof (Lip. case)

Since

$$X_t - X_t^{(n)} = \int_0^t \sigma(X_s) - \sigma(X_{\eta_n(s)}^{(n)})dW_s$$
$$= \int_0^t \sigma(X_s) - \sigma(X_s^{(n)})dW_s + \int_0^t \sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})dW_s.$$

if σ is Lipschitz continuous we have,

$$\mathbb{E}[|X_t - X_t^{(n)}|^2]$$
$$\leq 2\int_0^t \mathbb{E}[|\sigma(X_s) - \sigma(X_s^{(n)})|^2]ds + 2\int_0^t \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2]ds$$
$$\leq C\int_0^t \mathbb{E}[|X_s - X_s^{(n)}|^2]ds + C\int_0^t \mathbb{E}[|X_s^{(n)} - X_{\eta(s)}^{(n)}|^2]ds$$
$$\leq C\int_0^t \mathbb{E}[|X_s - X_s^{(n)}|^2]ds + \frac{C}{n}.$$

By Gronwall's inequality, we conclude

$$\mathbb{E}[|X_t - X_t^{(n)}|^2]^{1/2} \leq \frac{C}{n^{1/2}}.$$

However, if σ is NOT Lipschitz conti., we cannot use Gronwall's inequality.
We must consider the following differences:

(i)

$$|\sigma(X_s) - \sigma(X_s^{(n)})|$$

⇝ We CAN use the proof of Le Gall (Yamada and Watanabe approximation argument).

(ii)

$$|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|$$

⇝ We CANNOT use the proof of Le Gall.
⇝ We need to consider new idea.

GOAL

GOAL: FIND some $\alpha \in (0,1)$ such that

$$\int_0^T \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2]ds \leq \frac{C}{n^\alpha}.$$

Remark 2

If σ is bdd, UE and Hölder continuous, Lemaire and Menozzi (2010)[10] *prove the density of $X_t^{(n)}$ satisfy the Gaussian two sided bounded:*

$$C^{-1} g_{c^{-1}t}(x_0, y) \leq p_t^{(n)}(x_0, y) \leq C g_{ct}(x_0, y),$$

by using the parametrix method. Using this, we can prove

$$\int_0^T \mathbb{E}[|f_\sigma(X_s^{(n)}) - f_\sigma(X_{\eta_n(s)}^{(n)})|]ds \leq \frac{C}{n^{1/2}}.$$

However, if σ is NOT Hölder continuous, it is difficult to prove the Gaussian two sided bounded.

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[10] On some Non-Asymptotic Bounds for the Euler Scheme. Electron J. Probab., 15, 1645-1681

Tightness

New idea of the proof is “tightness" of the Euler-Maruyama scheme.

Lemma 1

Suppose σ *: m'ble and* $\underline{\sigma} \leq \sigma \leq \overline{\sigma}$. *Let* c_4*: const. of BDG ineq. For any* $\varepsilon, \chi > 0$ *with* $\delta := \frac{\chi \varepsilon^4}{c_4 \overline{\sigma}^4} \leq T$, *we have*

$$\mathbb{P}\left(\sup_{t \leq s \leq t+\delta} |X_s^{(n)} - X_t^{(n)}| \geq \varepsilon \right) \leq \delta \chi, \tag{2}$$

for any $t \in [0, T]$ *and* $n \in \mathbb{N}$.

Remark 3

(2) $\Rightarrow (X^{(n)})_{n \in \mathbb{N}}$ *: tight in* $C[0, T]$, *that is, for any* $\varepsilon > 0$, *there exists a compact set* $K \subset C[0, T]$ *such that for any* $n \in \mathbb{N}$,

$$\mathbb{P} \circ (X_{\cdot}^{(n)})^{-1}(K) \geq 1 - \varepsilon,$$

(e.g. Billingsley, Theorem 8.3).

Tightness

Corollary 1

Let $(\gamma_n)_n$ be a decreasing seq. s.t. $\gamma_n \in (0,1]$ and $\gamma_n \downarrow 0$ and $\gamma_n n^2 \to \infty$. Define

$$\varepsilon_n := \frac{\widetilde{c}}{\gamma_n^{1/4} n^{1/2}},\ \widetilde{c} := T^{1/2} c_4^{1/4} \overline{\sigma},\ \chi_n := \frac{\gamma_n n}{T},\ \delta_n := \frac{\chi_n \varepsilon_n^4}{c_4 \overline{\sigma}^4} = \frac{T}{n} \leq T,$$

and for each $k = 1, \ldots, n-1$,

$$\Omega_{k,n} := \left\{ \omega \in \Omega\ ;\ \sup_{\frac{kT}{n} \leq s \leq \frac{(k+1)T}{n}} |X_s^{(n)} - X_{\frac{kT}{n}}^{(n)}| \geq \varepsilon_n \right\}.$$

Then

$$\mathbb{P}(\Omega_{k,n}) \leq \delta_n \chi_n = \gamma_n.$$

Key lemma

Lemma 2

Suppose Ass. 2 holds. ($\sigma = \rho \circ (f_1 - f_2)$, $\underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}$, discontinuous points of σ are finite.). Then,

$$\int_0^T \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2]ds \leq \frac{C}{n^{2/5}}.$$

Proof: Using Corollary 1,

$$\begin{aligned}
&\int_0^T \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2]ds \\
&= \sum_{k=0}^{n-1} \int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}} \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2\{\mathbf{1}_{\Omega_{k,n}} + \mathbf{1}_{\Omega_{k,n}^c}\}]ds \\
&\leq 4\overline{\sigma}^2 \sum_{k=0}^{n-1} \int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}} \mathbb{P}(\Omega_{n,k})ds + \sum_{k=0}^{n-1} \int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}} A_s^{n,k} ds \\
&\leq 4\overline{\sigma}^2 T\gamma_n + \sum_{k=0}^{n-1} \int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}} A_s^{n,k} ds.
\end{aligned}$$

$$
\begin{aligned}
A_s^{n,k} &= \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2 \mathbf{1}_{\Omega_{k,n}^c}] \\
&= \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2 \mathbf{1}_{\Omega_{k,n}^c}\{\mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(\sigma)} + \mathbf{1}_{X_s^{(n)} \notin S^{\varepsilon_n}(\sigma)}\}] \\
&=: A_s^{n,k,1} + A_s^{n,k,2},
\end{aligned}
$$

where for the discontinuous points of σ denoted by $S(\sigma) := \{a_1^\sigma, \ldots, a_m^\sigma\}$,

$$
S^{\varepsilon_n}(\sigma) := \bigcup_{i=1}^{m} [a_i^\sigma - \varepsilon_n,\ a_i^\sigma + \varepsilon_n].
$$

$A_s^{n,k,2}$: On the set $\Omega_{k,n}^c \cap \{X_s^{(n)} \notin S^{\varepsilon_n}(\sigma)\}$, we have

$$
S(\sigma) \cap [X_s^{(n)} \wedge X_{\frac{kT}{n}}^{(n)}, X_s^{(n)} \vee X_{\frac{kT}{n}}^{(n)}] = \emptyset,
$$

thus, since $\sigma = \rho \circ f$ is "picewise" $1/2$-Hölder conti.

$$
|\sigma(X_s^{(n)}) - \sigma(X_{\frac{kT}{n}}^{(n)})|^2 \leq ||\sigma||_{\ell oc,1/2}^2 |X_s^{(n)} - X_{\frac{kT}{n}}^{(n)}|.
$$

Hence

$$
\sum_{k=0}^{n-1} \int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}} A_s^{n,k,2} ds \leq C \int_0^T \mathbb{E}[|X_s^{(n)} - X_{\eta_n(s)}^{(n)}|] ds \leq \frac{C}{n^{1/2}}.
$$

$A_s^{n,k,1}$: Recall that

$$A_s^{n,k,1} = \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2 \mathbf{1}_{\Omega_{k,n}^c} \mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(\sigma)}] \leq 4\overline{\sigma}^2 \mathbb{E}[\mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(\sigma)}].$$

Thus,

$$\sum_{k=0}^{n-1} \int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}} A_s^{n,k,1} ds \leq 4\overline{\sigma}^2 \mathbb{E}[\int_0^T \mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(\sigma)} ds]. \tag{3}$$

Since σ is uniformly positive,

$$\langle X^{(n)} \rangle_t = \int_0^t |\sigma(X_{\eta_n(s)}^{(n)})|^2 ds \geq \underline{\sigma}^2 t.$$

Hence by the occupation time formula, (3) is bounded by

$$4\overline{\sigma}^2 \underline{\sigma}^{-2} \mathbb{E}[\int_0^T \mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(\sigma)} d\langle X^{(n)} \rangle_s] = 4\overline{\sigma}^2 \underline{\sigma}^{-2} \mathbb{E}[\int_{\mathbb{R}} \mathbf{1}_{x \in S^{\varepsilon_n}(\sigma)} L_T^x(X^{(n)}) dx]$$

$$= 4\overline{\sigma}^2 \underline{\sigma}^{-2} \int_{S^{\varepsilon_n}(\sigma)} \mathbb{E}[L_T^x(X^{(n)})] dx$$

$$\leq C Leb(S^{\varepsilon_n}(\sigma)), \qquad (\because \sup_{n \in \mathbb{N}, x \in \mathbb{R}} \mathbb{E}[L_T^x(X^{(n)})] < \infty),$$

$$= C \sum_{i=1}^{m} Leb([a_i^\sigma - \varepsilon_n,\ a_i^\sigma + \varepsilon_n]) = 2Cm\varepsilon_n.$$

Therefore, we conclude

$$\int_0^T \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2]ds \leq C\left\{\gamma_n + \frac{1}{n^{1/2}} + \varepsilon_n\right\}.$$

By choosing $\gamma_n := \frac{1}{n^{2/5}}$, and then

$$\varepsilon_n = \frac{\widetilde{c}}{\gamma_n^{1/4} n^{1/2}} = \frac{\widetilde{c}}{n^{2/5}}.$$

Therefore, we obtain

$$\int_0^T \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2]ds \leq C\left\{\frac{1+\widetilde{c}}{n^{2/5}} + \frac{1}{n^{1/2}}\right\} \leq \frac{C(2+\widetilde{c})}{n^{2/5}}.$$

This concludes the proof. $\square$

Yamada-Watanabe approximation technique

Yamada-Watanabe approximation technique

For each $\delta \in (1, \infty)$ and $\varepsilon \in (0, 1)$, we define a continuous function $\psi_{\delta,\varepsilon} : \mathbb{R} \to \mathbb{R}^+$ with $supp\, \psi_{\delta,\varepsilon} \subset [\varepsilon/\delta, \varepsilon]$ such that

$$\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta,\varepsilon}(z)dz = 1 \text{ and } 0 \le \psi_{\delta,\varepsilon}(z) \le \frac{2}{z \log \delta}, \quad z > 0.$$

Since $\int_{\varepsilon/\delta}^{\varepsilon} \frac{2}{z \log \delta} dz = 2$, there exists such a function $\psi_{\delta,\varepsilon}$. We define a function $\phi_{\delta,\varepsilon} \in C^2(\mathbb{R}; \mathbb{R})$ by

$$\phi_{\delta,\varepsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta,\varepsilon}(z)dzdy.$$

It is easy to verify that $\phi_{\delta,\varepsilon}$ has the following useful properties:

$$|x| \le \varepsilon + \phi_{\delta,\varepsilon}(x), \text{ for any } x \in \mathbb{R}, \tag{4}$$

$$0 \le |\phi'_{\delta,\varepsilon}(x)| \le 1, \text{ for any } x \in \mathbb{R}, \tag{5}$$

$$\phi''_{\delta,\varepsilon}(\pm|x|) = \psi_{\delta,\varepsilon}(|x|) \le \frac{2}{|x| \log \delta} \mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|x|), \text{ for any } x \in \mathbb{R} \setminus \{0\}. \tag{6}$$

Proof of Theorem 7

From (4), for any $t \in [0, T]$, we have

$$|X_t - X_t^{(n)}| \le \varepsilon + \phi_{\delta,\varepsilon}(X_t - X_t^{(n)}). \tag{7}$$

Using Itô's formula, we have

$$\phi_{\delta,\varepsilon}(X_t - X_t^{(n)}) = M_t^{n,\delta,\varepsilon} + J_t^{n,\delta,\varepsilon}, \tag{8}$$

where

$$M_t^{n,\delta,\varepsilon} := \int_0^t \phi'_{\delta,\varepsilon}(X_s - X_s^{(n)}) \left\{\sigma(X_s) - \sigma(X_{\eta_n(s)}^{(n)})\right\} dW_s,$$

$$J_t^{n,\delta,\varepsilon} := \frac{1}{2}\int_0^t \phi''_{\delta,\varepsilon}(X_s - X_s^{(n)})|\sigma(X_s) - \sigma(X_{\eta_n(s)}^{(n)})|^2 ds.$$

Since $\phi'_{\delta,\varepsilon}$ and σ are bounded, $M^{n,\delta,\varepsilon}$ is martingale hence $\mathbb{E}[M_t^{n,\delta,\varepsilon}] = 0$.

Proof of Theorem 7

Using (6)

$$\begin{aligned} J_t^{n,\delta,\varepsilon} &= \frac{1}{2}\int_0^t \phi''_{\delta,\varepsilon}(X_s - X_s^{(n)})|\sigma(X_s) - \sigma(X_{\eta_n(s)}^{(n)})|^2 ds \\ &\leq 2\int_0^T \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|X_s - X_s^{(n)}|)}{|X_s - X_s^{(n)}|\log\delta}\{|\sigma(X_s) - \sigma(X_s^{(n)})|^2 + |\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2\} ds \\ &=: J_t^{n,\delta,\varepsilon,1} + J_t^{n,\delta,\varepsilon,2}. \end{aligned}$$

Using the Assumption 1, approximation argument, IBP and estimation of local time, we have

$$J_t^{n,\delta,\varepsilon,1} \leq \frac{C}{\log\delta}. \qquad (9)$$

Using Lemma 2, we have

$$\mathbb{E}[J_t^{n,\delta,\varepsilon,2}] \leq \frac{2\delta}{\varepsilon\log\delta}\int_0^T \mathbb{E}[|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2] ds \leq \frac{C\delta}{\varepsilon\log\delta}\frac{1}{n^{2/5}}. \qquad (10)$$

It follows from (7), (8), (9) and (10) that

$$\sup_{0\le t\le T} \mathbb{E}[|X_t - X_t^{(n)}|] \le \varepsilon + \frac{C}{\log\delta} + \frac{C\delta}{\varepsilon\log\delta}\frac{1}{n^{2/5}}$$

for any $\varepsilon \in (0,1)$ and $\delta \in (1,\infty)$. By choosing $\varepsilon = \frac{1}{\log n}$ and $\delta = n^{1/5}$, we obtain

$$\begin{aligned}\sup_{0\le t\le T} \mathbb{E}[|X_t - X_t^{(n)}|] &\le \frac{1}{\log n} + \frac{C}{\frac{1}{5}\log n} + \frac{Cn^{1/5}}{\frac{1}{\log n}\frac{1}{5}\log n}\frac{1}{n^{2/5}} \\ &\le \frac{C}{\log n} + \frac{C}{n^{1/5}}.\end{aligned}$$

This concludes the proof. □

Remark on degenerate case

Theorem 8 (Hairer, Hutzenthaler and Jentzen (2015)[11])

Let X be a solution of 4-dimensional SDE $dX_t = \mu(X_t)dt + BdW_t$ with

$$\mu(x) = \begin{pmatrix} \mathbf{1}_{(1,\infty)}(x_4)\exp\left(-\frac{1}{x_4^2-1}\right)\cos\left((x_3-\hat{C})\cdot\exp(x_2^3)\right) \\ 0 \\ \mathbf{1}_{(-1,1)}(x_4)\exp\left(-\frac{1}{1-x_4^2}\right) \\ 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\hat{C} := \int_0^1 e^{-1/(1-x^2)}dx$. *If* $X_0 = 0$, *then for any* $\alpha \in [0,\infty)$,

$$\lim_{n\to\infty} n^{\alpha}\mathbb{E}[|X_t - X_t^{(n)}|] = \lim_{n\to\infty} n^{\alpha}|\mathbb{E}[X_t] - \mathbb{E}[X_t^{(n)}]| = \begin{cases} 0 & \text{if } \alpha = 0, \\ \infty & \text{if } \alpha > 0. \end{cases}$$

Remark 4

Leobacher and Szölgyenyi (2016)[12] prove that by using the same argument of Cor. 1, the L^2-conv. rate is $1/5$ when the drift is picewise Lipschitz and diffusion coefficient is Lip. conti. and degenerate.

[11] Loss of regularity for Kolmogorov equation. Ann. Probab. 43(2), 468-527

[12] Convergence of the Euler-Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient, arXiv:1610.07047.