On the Euler-Maruyama scheme for SDEs with discontinuous diffusion coefficient

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Abstract

In this talk, we consider the strong rate of convergence for the Euler-Maruyama scheme of a class of stochastic differential equations whose diffusion coefficient is discontinuous.

Euler-Maruyama scheme

Let $X = (X_t)_{0 \leq t \leq T}$ be the solution of the one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t \sigma(X_s) dW_s, \ x_0 \in \mathbb{R}, \ t \in [0, T], \tag{1}$$

where $W = (W_t)_{0 \leq t \leq T}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions.

The solution of (1) is rarely analytically tractable, so one often approximates X by using the Euler-Maruyama approximation $X^{(n)} = (X_t^{(n)})_{0 \leq t \leq T}$ given by

$$X_t^{(n)} = x_0 + \int_0^t \sigma(X_{\eta_n(s)}^{(n)}) dW_s, \ t \in [0, T],$$

where $\eta_n(s) = kT/n$ if $s \in [kT/n, (k+1)T/n)$. It is well-known that if σ is Lipschitz continuous, the Euler-Maruyama approximation for (1) converges at the strong rate of order $1/2$, that is, there exists $C > 0$ such that

$$\mathbb{E}[|X_T - X_T^{(n)}|] \leq \frac{C}{n^{1/2}}.$$

The strong rate in the case of non-Lipschitz coefficient has been studied recently. Gyöngy and Rásonyi [1] prove that if σ is α-Hölder continuous with $\alpha \in [1/2, 1]$, then there exists $C > 0$ such that

$$\mathbb{E}[|X_T - X_T^{(n)}|] \leq \begin{cases} \dfrac{C}{n^{\alpha - 1/2}} & \text{if } \alpha \in (1/2, 1], \\ \dfrac{C}{\log n} & \text{if } \alpha = 1/2. \end{cases}$$

These results still hold for the SDE with discontinuous drift coefficient ([3]).

SDE with discontinuous diffusion coefficient

In this talk, we assume that the diffusion coefficient σ satisfies the following condition:

$$\sigma := \rho \circ f,$$

where ρ is 1/2-Hölder continuous and there exists $0 < \underline{\rho} < \overline{\rho}$ such that

$$\underline{\rho} \leq \rho(x) \leq \overline{\rho},$$

and $f = f_1 - f_2$, f_1 and f_2 are bounded, strictly increasing with finite discontinuous points. Note that under the above assumption, the SDE (1) has a unique strong solution, (see [2]).

In this talk, under the above assumption for the diffusion coefficient σ, we will show that the Euler-Maruyama approximation $X^{(n)}$ converges to the unique solution to the corresponding SDE in L^1-sense with the rate $\log n$, that is there exists $C > 0$ such that for any $n \geq 2$,

$$\mathbb{E}[|X_T - X_T^{(n)}|] \leq \frac{C}{\log n}.$$

The idea of proof is to use the "tightness" and some estimations of the local time of the Euler-Maruyama approximation.

References

[1] Gyöngy, I. and Rásonyi, M.: A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients. Stochastic. Process. Appl. 121, 2189–2200 (2011).

[2] Le Gall, JF.: One-dimensional stochastic differential equations involving the local times of the unknown process. In Stochastic analysis and applications 1984 (pp. 51-82). Springer Berlin Heidelberg.

[3] Ngo, H-L., and Taguchi, D.: Strong rate of convergence for the Euler-Maruyama approximation of stochastic differential equations with irregular coefficients. Math. Comp. 85(300), 1793–1819 (2016).

[4] Ngo, H-L., and Taguchi, D.: Strong convergence for the Euler-Maruyama approximation of stochastic differential equations with discontinuous coefficients. Preprint, arXiv:1604.01174v2.