Stochastic complex Ginzburg-Landau equation with space-time white noise

(Joint work with Dr. Masato Hoshino and Prof. Yuzuru Inahama)

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$$\begin{cases} \partial_t u = (\mathrm{i} + \mu)\triangle u + \nu(1 - |u|^2)u + \xi \\ \qquad\qquad\qquad\qquad \text{on } (0, \infty) \times \mathbf{T}^3, \\ u(0, \cdot) = u_0(\cdot), \end{cases}$$

where

- $\mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$: 3-dim. torus,
- ξ: $\mathbf{C}$-val. space-time white noise,
- $\mathrm{i} = \sqrt{-1}$,
- μ is a positive constant,
- ν is a complex constant.

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The white noise is too rough to define $|u|^2 u$.

Notation

- $\mathcal{D}$: the space of all smooth functions on $\mathbf{T}^3$,
- $\mathcal{D}'$: the dual of $\mathcal{D}$,
- $\{\mathbf{e}_k\}_{k\in\mathbf{Z}^3}$: CONS of $L^2(\mathbf{T}^3)$, i.e. $\mathbf{e}_k(x) = e^{2\pi i k\cdot x}$,
- $\{\rho_m\}_{m=-1}^{\infty}$: dyadic partition of unity,

(1) $\rho_m \colon \mathbf{R} \to [0,1]$, radial, smooth,

(2) $\mathrm{supp}(\rho_{-1}) \subset B(0, \frac{4}{3})$,
$\mathrm{supp}(\rho_0) \subset B(0, \frac{8}{3}) \setminus B(0, \frac{4}{3})$,

(3) $\rho_m(\cdot) = \rho_0(2^{-m}\cdot)$,

(4) $\sum_{m=-1}^{\infty} \rho_m(\cdot) = 1$,

Notation

- $\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbf{T}^3} \mathbf{e}_{-k}(x) f(x)\, dx$ for $k \in \mathbf{Z}^3$,
- $\mathcal{F}^{-1}\phi(x) = \sum_{k\in\mathbf{Z}^3} \phi(k)\mathbf{e}_k(x)$ for $x \in \mathbf{T}^3$
- $\phi(D)f = \mathcal{F}^{-1}\phi\mathcal{F}f = \sum_{k\in\mathbf{Z}^3} \phi(k)\hat{f}(k)\mathbf{e}_k$,
- $\triangle_m = \rho_m(D)$,
- $\mathcal{C}^\alpha$: the Besov-Hölder space with the Hölder exponent $\alpha \in \mathbf{R}$, i.e. the completion of $C^\infty(\mathbf{T}^3, \mathbf{C})$ under the norm

$$\|f\|_{\mathcal{C}^\alpha} = \sup_{m\geqslant -1} 2^{m\alpha}\|\triangle_m f\|_{L^\infty}$$

$\alpha + \beta > 0$ is necessary to define *fg* !

For every $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$, a formal calc. implies

$$\begin{aligned} fg &= \Big(\sum \triangle_{m_1} f\Big)\Big(\sum \triangle_{m_2} g\Big) \\ &= \Bigg(\sum_{m_1 \geqslant m_2+2} + \sum_{|m_1-m_2|\leqslant 1} + \sum_{m_1+2\leqslant m_2}\Bigg) \triangle_{m_1} f \triangle_{m_2} g \\ &=: f \varolessthan g + f \odot g + f \varogreaterthan g \end{aligned}$$

$\alpha + \beta > 0$ is necessary to define *fg* !

Theorem (Bony '81)

(1) $\forall \alpha, \forall \beta$, $\|f \varolessthan g\|_{\mathcal{C}^{\alpha \wedge 0 + \beta}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}$.

(2) *If* $\alpha + \beta > 0$, *then* $\|f \odot g\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}$.

If and only if $\alpha + \beta > 0$, we can define

$$fg = \underbrace{f \varogreaterthan g}_{\mathcal{C}^{\beta \wedge 0 + \alpha}} + \underbrace{f \odot g}_{\mathcal{C}^{\alpha+\beta}} + \underbrace{f \varolessthan g}_{\mathcal{C}^{\alpha \wedge 0 + \beta}} \in \mathcal{C}^{\alpha \wedge \beta \wedge (\alpha+\beta)}.$$

Commutator estimate

Proposition

Let $0 < \alpha < 1$, $\beta, \gamma \in \mathbf{R}$ *satisfy* $\beta + \gamma < 0$ *and* $\alpha + \beta + \gamma > 0$. *For* $f, g, h \in C^\infty(\mathbf{T}^3, \mathbf{C})$, *we set*

$$R(f, g, h) = (f \varolessthan g) \odot h - f(g \odot h).$$

Then R is extended to a conti. trilinear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ *to* $\mathcal{C}^{\alpha+\beta+\gamma}$ *uniquely.*

From this proposition, we obtain

$$\begin{aligned} fgh = {} & f(g \odot h) + (f \odot g)h + R(f, g, h) \\ & + (f \varolessthan g)(\varolessthan + \varogreaterthan)h + (f \varogreaterthan g)h. \end{aligned}$$

Simplified Schauder estimate

- $\mathcal{L}^1 = \partial_t - \{(\mathrm{i}+\mu)\triangle - 1\}$,
- $P^1_t = e^{t\{(\mathrm{i}+\mu)\triangle-1\}}$ and $I(u)_t = \int_0^t P^1_{t-s} u_s \, ds$,
- $C_T\mathcal{C}^\alpha = C([0,T],\mathcal{C}^\alpha)$ which is equipped with the supremum norm $\|\cdot\|_{C_T\mathcal{C}^\alpha}$.

Proposition

For every $u \in L^\infty_T\mathcal{C}^\alpha$ and $\gamma \in [\alpha, \alpha+2)$, we have

$$\|I(u)\|_{C_T\mathcal{C}^\gamma} \lesssim T^{\frac{\alpha+2-\gamma}{2}} \|u\|_{L^\infty_T\mathcal{C}^\alpha}.$$

The white noise is too rough to define $|u|^2 u$

- Note $\mathcal{L}^1 u = -\nu u^2 \overline{u} + (\nu + 1)u + \xi$.
- Consider the linearized eq. $\mathcal{L}^1 Z = \xi$.
  From $\xi \in L_T^\infty \mathcal{C}^{-\frac{5}{2}-}$ and the Schauder est., the sol. $Z = I(\xi)$ satisfies

$$Z \in C_T \mathcal{C}^{-\frac{1}{2}-}.$$

- Since the irregularity of u comes from ξ, it is natural to guess

$$u \in C_T \mathcal{C}^{-\frac{1}{2}-}.$$

- The product $|u_t|^2 u_t = u_t^2 \overline{u_t}$ is not defined.

Main result

- Let $\{\rho^\epsilon\}_{0<\epsilon<1}$ be a mollifier on $\mathbf{T}^3$,
- Set $\xi^\epsilon = \rho^\epsilon * \xi$ for every $0 < \epsilon < 1$,
- Take some const. $\mathfrak{c}^\epsilon$, which $\mathfrak{c}^\epsilon \to \infty$ as $\epsilon \downarrow 0$,
- $0 < \kappa < \kappa' \ll 1$

Theorem (Hoshino-Inahama-N.)

Let $u_0 \in \mathcal{C}^{-\frac{2}{3}+\kappa'}$. Consider the renormalized eq.

$$\begin{cases} \quad \partial_t u^\epsilon = (\mathrm{i} + \mu)\triangle u^\epsilon + \nu(1 - |u^\epsilon|^2)u^\epsilon \\ \qquad\qquad\qquad\qquad + \nu\mathfrak{c}^\epsilon u^\epsilon + \xi^\epsilon, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

Theorem (Hoshino-Inahama-N.)

There exist a unique proc. u^ϵ *and a random time* T_*^ϵ *s.t.*

- u^ϵ *solves the eq. on* $[0, T_*^\epsilon) \times \mathbf{T}^3$,
- T_*^ϵ *converges to some a.s. positive random time* T_* *in probability,*
- u^ϵ *converges to some proc.* u *defined on* $[0, T_*) \times \mathbf{T}^3$ *in the sense that* $u^\epsilon \to u$ *in probability in* $C_t \mathcal{C}^{-\frac{3}{2}+\kappa'}$ *for every* $0 < t < T_*$. *Furthermore,* u *is indep. of the choice of* ξ^ϵ.

Strategy

- We discuss our problem as deterministic one.
- We take deterministic $\xi \in C_T\mathcal{C}^{-\frac{5}{2}-}$ and fix it. Note $Z = I(\xi) \in C_T\mathcal{C}^{-\frac{1}{2}-}$.
- We construct
  - the product $u^2\overline{u}$ by focusing on the most rough term in nonlinear terms,
  - a solution map from $\mathcal{X}_T^{\kappa}$ to $\mathcal{D}_T^{\kappa,\kappa'}$.

Rule and Notation

- We use the rule that

$$f \in \mathcal{C}^{\alpha}, g \in \mathcal{C}^{\beta} \Longrightarrow fg \in \mathcal{C}^{\alpha \wedge \beta \wedge (\alpha+\beta)}$$

for any $\alpha, \beta \in \mathbf{R}$.

- We use the tree-like symbols:

 $\rightsquigarrow Z = I(\xi)$,  $\rightsquigarrow \overline{Z}$,  $\rightsquigarrow Z^2$,  $\rightsquigarrow Z\overline{Z}$.

In this symbols,

dot $\rightsquigarrow$ the white noise,

line $\rightsquigarrow$ the operation I.

Reduction of our problem (Step 1)

Recall

$$\mathcal{L}^1 u = \underbrace{-\nu \underbrace{u^2\overline{u}}_{-\frac{3}{2}-} + (\nu+1)\underbrace{u}_{-\frac{1}{2}-}}_{-\frac{3}{2}-} + \underbrace{\xi}_{-\frac{5}{2}-}.$$

We set $u = u_1 + Z$ for $u_1 \in C_T\mathcal{C}^{\frac{1}{2}-}$ $(-\frac{3}{2} + 2 = \frac{1}{2})$.
Then

$$\mathcal{L}^1 u_1 = -\nu(u_1 + Z)^2(\overline{u_1} + \overline{Z}) + (\nu+1)(u_1 + Z).$$

Reduction of our problem (Step 2)

Note that the term $(u_1 + Z)^2(\overline{u_1} + \overline{Z})$ is eqaul to

$$\underbrace{\underbrace{u_1^2 u_1}_{+\frac{1}{2}-} + \underbrace{u_1^2\overline{Z}}_{0-} + 2\,\underbrace{u_1\overline{u_1}Z}_{0-} + 2\,\underbrace{u_1\,\underbrace{Z\overline{Z}}_{-1-}}_{-1-} + \underbrace{\overline{u_1}\,\underbrace{Z^2}_{-1-}}_{-1-}}_{-1-} + \underbrace{Z^2\overline{Z}}_{-\frac{3}{2}-}$$

- We replace $Z\overline{Z}$, Z^2 and $Z^2\overline{Z}$ by $X^{\vee}, X^{\Vee} \in C_T\mathcal{C}^{-1-}$ and $X^{\Ydown} \in C_T\mathcal{C}^{-\frac{3}{2}-}$, respectively, because they are ill-posed.
- We introduce $W \in C_T\mathcal{C}^{\frac{1}{2}-}$ s.t. $\mathcal{L}^1 W = X^{\Ydown}$ and set $u_1 = u_2 - \nu W$ for $u_2 \in C_T\mathcal{C}^{1-}$.

Then

$$(*)\quad \mathcal{L}^1 u_2 = -\nu\{(u_2-\nu W)^2(\overline{u_2}-\overline{\nu}\overline{W}) + (u_2-\nu W)^2\overline{Z}$$

$$+\,2(u_2-\nu W)(\overline{u_2}-\overline{\nu}\overline{W})Z$$

$$+\,2(u_2-\nu W)X^{\bullet\vee\circ} + (\overline{u_2}-\overline{\nu}\overline{W})X^{\bullet\vee\bullet}\}$$

$$+\,(\nu+1)(u_2-\nu W+Z)$$

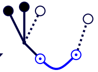
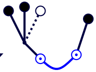
$$= -\nu\{2(u_2-\nu W)X^{\bullet\vee\circ}+(\overline{u_2}-\overline{\nu}\overline{W})X^{\bullet\vee\bullet}\}+G_{-1}(u_2).$$

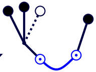
In (RHS), the following terms are ill-defined:

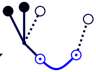
$$u_2 W\overline{Z},\quad W^2\overline{Z},\quad u_2\overline{W}Z,\quad \overline{u_2}WZ,\quad W\overline{W}Z,$$

$$u_2 X^{\bullet\vee\circ},\quad WX^{\bullet\vee\circ},\quad \overline{u_2}X^{\bullet\vee\bullet},\quad \overline{W}X^{\bullet\vee\bullet}.$$

Definitions of $W\overline{Z}$, $W^2\overline{Z}$, $\overline{W}Z$, WZ and $W\overline{W}Z$

- We introduce X , X  $\in C_T\mathcal{C}^{0-}$ and define $WZ, W\overline{Z}, W^2\overline{Z}, W\overline{W}Z \in C_T\mathcal{C}^{-\frac{1}{2}-}$.
- From $u_2 \in C_T\mathcal{C}^{1-}$, the products $u_2 W\overline{Z}$, $u_2\overline{W}Z$ and $\overline{u_2}WZ$ are well-defined.
- In fact

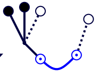
$$WZ = W(\olessthan + \ogreaterthan)Z + X$$  ,

$$W\overline{Z} = W(\olessthan + \ogreaterthan)Z + X$$  .

Definitions of $W\overline{Z}$, $W^2\overline{Z}$, $\overline{W}Z$, WZ and $W\overline{W}Z$

- From the commutator estimate, we can define

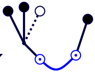
$$W^2\overline{Z} = W\overline{Z}W$$

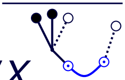
$$= 2WX$$


$$+ R(W, \overline{Z}, W)$$

$$+ (W \oslash \overline{Z})(\oslash + \oslash)W + W(W \oslash \overline{Z}),$$

$$W\overline{W}Z = \overline{W}ZW$$

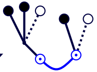
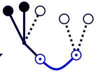
$$= \overline{W}X$$


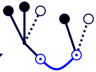
$$+ WX$$


$$+ R(\overline{W}, Z, W)$$

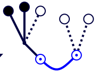
$$+ (\overline{W} \oslash Z)(\oslash + \oslash)W + W(\overline{W} \oslash Z).$$

Definitions WX and $\overline{W}X$

- We introduce X, X $\in C_T\mathcal{C}^{-1-}$ and define

WX $= W(\otimes + \otimes)X$ $+ X$,

$\overline{W}X$ $= \overline{WX}$ (with the overline also covering the tree)

$= \overline{W(\otimes + \otimes)X}$ $\overline{+ X}$ (a single overline covers the whole of this line).

We cannot define $u_2 X^{\vee}$ and $\overline{u_2} X^{\vee}$

- Since $u_2 \in C_T \mathcal{C}^{1-}$ and $X^{\vee}, X^{\vee} \in C_T \mathcal{C}^{-1-}$, we cannot define $u_2 X^{\vee}$ and $\overline{u_2} X^{\vee}$.
- The replacement $u_2 = u_3 +$ (proc.) does not work well. Because $u_3 \in C_T \mathcal{C}^{1-}$.
- We need new idea.

Reduction of our problem (Step 3)

We consider the following system:

$$\mathcal{L}^1 v = F(v, w), \quad \mathcal{L}^1 w = G_0(v, w) + G_{-1}(v + w),$$

where

$$\begin{aligned} F(v, w) = -\nu\{ & 2(v + w - \nu W) \varolessthan X^{\bullet\vee\circ} \\ & + (\overline{v} + \overline{w} - \overline{\nu}\overline{W}) \varolessthan X^{\bullet\vee\bullet}\}, \\ G_0(v, w) = -\nu\{ & 2(v + w - \nu W)(\varogreaterthan + \odot) X^{\bullet\vee\circ} \\ & + (\overline{v} + \overline{w} - \overline{\nu}\overline{W})(\varogreaterthan + \odot) X^{\bullet\vee\bullet}\} \end{aligned}$$

Then $u_2 = v + w \in C_T\mathcal{C}^{1-}$ solves $(*)$.

Regularity of v and ill-posedness of $v \odot X^{\vee\!\!\circ}$

- $F(v, w) \in C_T\mathcal{C}^{-1-}$ follows from
  - $v + w \in C_T\mathcal{C}^{1-}$, $W \in C_T\mathcal{C}^{\frac{1}{2}-}$,
  - $X^{\vee\!\!\circ}, X^{\vee} \in C_T\mathcal{C}^{-1-}$.
- Hence, $v \in C_T\mathcal{C}^{+1-}$.
- The resonants $v \odot X^{\vee\!\!\circ}$, $\overline{v} \odot X^{\vee}$ are ill-posed.

Definition of $v \odot X^{\circ\bullet}$ and $\overline{v} \odot X^{\circ\circ}$

- In order to define them, we introduce
  - $X^{\bullet\circ} \in C_T\mathcal{C}^{+1-}$ s.t. $\mathcal{L}^1 X^{\bullet\circ} = X^{\bullet\circ}$,
  - $X^{\bullet\bullet} \in C_T\mathcal{C}^{+1-}$ s.t. $\mathcal{L}^1 X^{\bullet\bullet} = X^{\bullet\bullet}$,
  - $X^{\bullet\circ\,\bullet\circ}, X^{\bullet\bullet\,\bullet\circ}, X^{\bullet\circ\,\circ\circ}, X^{\bullet\bullet\,\circ\circ} \in C_T\mathcal{C}^{0-}$.
- Define $\mathrm{com}(v, w) \in C_T\mathcal{C}^{1+}$ by

$$v + \nu\{2(v + w - \nu W) \otimes X^{\bullet\circ}$$

$$+ (\overline{v} + \overline{w} - \overline{\nu}\overline{W}) \otimes X^{\bullet\bullet}\}.$$

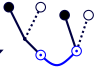
- Note that

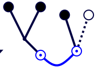
$$v = \mathrm{com}(v, w) - \nu\{2(v + w - \nu W) \olessthan X^{Y_{\bullet\circ}} + (\overline{v} + \overline{w} - \overline{\nu}\overline{W}) \olessthan X^{Y_{\bullet\bullet}}\}$$

and

$$((v + w - \nu W) \olessthan X^{Y_{\bullet\circ}}) \odot X^{V_{\bullet\circ}} = R((v + w - \nu W), X^{Y_{\bullet\circ}}, X^{V_{\bullet\circ}}) + (v + w - \nu W)(X^{Y_{\bullet\circ}} \odot X^{V_{\bullet\circ}}).$$

- $v \odot X$ is defined by

$$-\nu\{2(v + w - \nu W)X$$ 

$$+ (\overline{v} + \overline{w} - \overline{\nu}\overline{W})X$$ 

$$+ 2R(v + w - \nu W, X$$  $, X$  $)$

$$+ R(\overline{v} + \overline{w} - \overline{\nu}\overline{W}, X$$  $, X$  $)\}$

$$+ \mathrm{com}(v, w) \odot X$$  .

- $\overline{v} \odot X$ is defined in the similar way.

We can construct a conti. sol. map

For every $0 < \kappa < \kappa' < 1/18$ and $T > 0$, we set

$$\mathcal{X}_T^{\kappa} = C_T\mathcal{C}^{-\frac{1}{2}-\kappa} \times \cdots \ni X = (X^{\mathbf{I}}, \dots),$$
$$\mathcal{D}_T^{\kappa,\kappa'} \subset C_T\mathcal{C}^{-\frac{2}{3}+\kappa'} \times C_T\mathcal{C}^{-\frac{1}{2}-2\kappa}.$$

We call $X \in \mathcal{X}_T^{\kappa}$ driving vector.

Theorem

(1) *For every* $(v_0, w_0) \in \mathcal{C}^{-\frac{2}{3}+\kappa'} \times \mathcal{C}^{-\frac{1}{2}-2\kappa}$ *and* $X \in \mathcal{X}_T^{\kappa}$, *there exists* $T_* \in (0, 1]$ *such that the system admits a unique sol.* $(v, w) \in \mathcal{D}_{T_*}^{\kappa,\kappa'}$.

(2) *The map* $(v_0, w_0, X) \mapsto (v, w)$ *is conti.*

Construction of a driving vector X

We construct a driving vector $X = (X^{\text{¡}}, \dots)$ s.t.

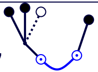
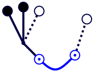
$$X^{\text{¡}} = I(\xi),$$

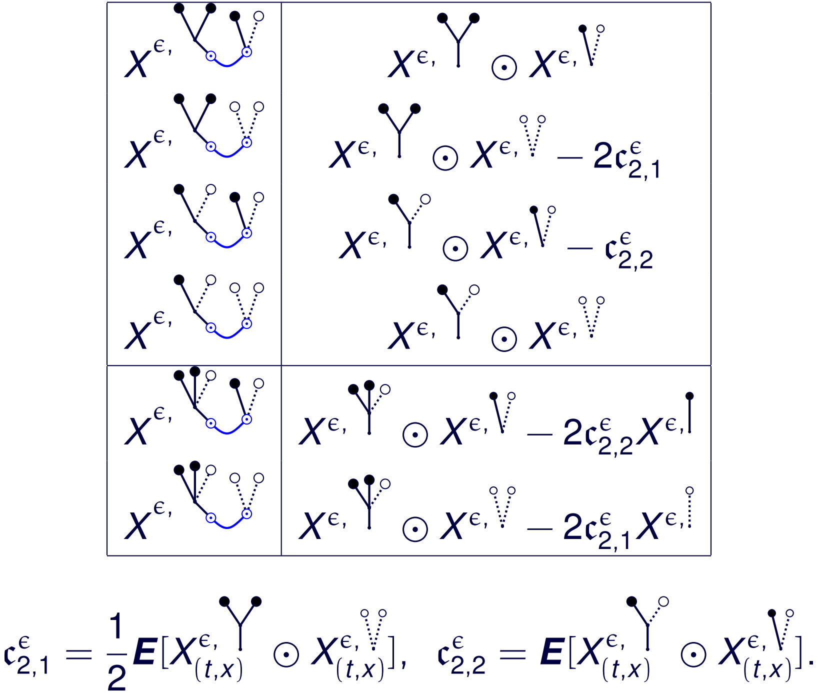
where $I(u) = P_t^1 \int_{-\infty}^0 P_{-s}^1 u_s \, ds + \int_0^t P_{t-s}^1 u_s \, ds$.
To this end, we set

- $\chi^\epsilon = \hat{\rho}^\epsilon$,
- $\xi_t^\epsilon = \rho^\epsilon * \xi_t$,
- $X^{\epsilon,\text{¡}} = Z^\epsilon = I(\xi^\epsilon)$,
- $\mathfrak{c}_1^\epsilon = \boldsymbol{E}[Z_{(t,x)}^\epsilon \overline{Z_{(t,x)}^\epsilon}]$: const. indep. of (t, x).

We define distributions $X^{\epsilon,\tau}$ as follows:

Symbol	Definition
$X^{\epsilon,\bullet}$	Z^{ϵ}
$X^{\epsilon,\circ}$	$\overline{X^{\epsilon,\bullet}}$
$X^{\epsilon,\bullet\bullet}$	$(X^{\epsilon,\bullet})^{2}$
$X^{\epsilon,\bullet\circ}$	$X^{\epsilon,\bullet}X^{\epsilon,\circ} - \mathfrak{c}_1^{\epsilon}$
$X^{\epsilon,\circ\circ}$	$(X^{\epsilon,\circ})^{2}$
$X^{\epsilon,\bullet\bullet\circ}$	$X^{\epsilon,\bullet\bullet}X^{\epsilon,\circ} - 2\mathfrak{c}_1^{\epsilon}X^{\epsilon,\bullet}$

Symbol	Definition
$X^{\epsilon,}$ 	$I(X^{\epsilon,}$  $)$
$X^{\epsilon,}$ 	$I(X^{\epsilon,}$  $)$
$X^{\epsilon,}$  $= W^{\epsilon}$	$I(X^{\epsilon,}$  $)$
$X^{\epsilon,}$ 	$X^{\epsilon,}$  $\odot X^{\epsilon,}$ 
$X^{\epsilon,}$ 	$X^{\epsilon,}$  $\odot X^{\epsilon,}$ 


$- 2\mathfrak{c}^{\epsilon}_{2,1}$
$- \mathfrak{c}^{\epsilon}_{2,2}$
$- 2\mathfrak{c}^{\epsilon}_{2,2}$
$- 2\mathfrak{c}^{\epsilon}_{2,1}$
$\mathfrak{c}^{\epsilon}_{2,1} = \frac{1}{2}\boldsymbol{E}[\ldots]$, $\mathfrak{c}^{\epsilon}_{2,2} = \boldsymbol{E}[\ldots]$.

Renormalized equation

For $X^\epsilon = (X^{\epsilon, \dot{|}}, \dots)$, we consider the system

$$\begin{cases} \mathcal{L}^1 v^\epsilon = F(v^\epsilon, w^\epsilon), \\ \mathcal{L}^1 w^\epsilon = G_0(v^\epsilon, w^\epsilon) + G_{-1}(v^\epsilon + w^\epsilon). \end{cases}$$

Then, $u^\epsilon = Z^\epsilon - \nu W^\epsilon + v^\epsilon + w^\epsilon$ solves

$$\partial_t u^\epsilon = (\mathrm{i} + \mu)\triangle u^\epsilon + \nu(1 - |u^\epsilon|^2)u^\epsilon + \nu \mathfrak{c}^\epsilon u^\epsilon + \xi^\epsilon.$$

with the initial condition $u_0^\epsilon = Z_0^\epsilon - \nu W_0^\epsilon + v_0^\epsilon + w_0^\epsilon$.
Here $\mathfrak{c}^\epsilon = 2(\mathfrak{c}_1^\epsilon - \overline{\nu \mathfrak{c}_{2,1}^\epsilon} - 2\nu \mathfrak{c}_{2,2}^\epsilon)$.

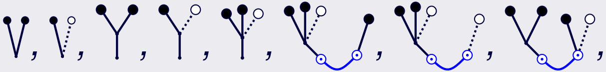
Convergence of X^{ϵ}

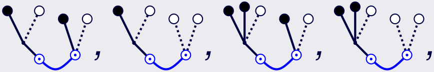
Let $\kappa > 0$, $T > 0$ and $1 < p < \infty$.

Proposition

$$\lim_{\epsilon \downarrow 0} \boldsymbol{E}\left[\|X^{\overset{\bullet}{|}} - X^{\epsilon, \overset{\bullet}{|}}\|^{p}_{C_T \mathcal{C}^{-1/2-\kappa}}\right] = 0.$$

Proposition

For $\tau =$ 

 , $\{X^{\epsilon,\tau}\}_{0<\epsilon<1}$ *is a Cauchy sequence in* $L^p(\Omega, C_T \mathcal{C}^{\alpha^{\tau}-\kappa})$*. Here,* α^{τ} *is suitable regularity.*

From these propositions, we see the following:

Theorem

$\exists \mathcal{X}_T^\kappa$-val process X s.t. $X^{\text{⫯}} = I(\xi)$ and

$$\lim_{\epsilon \downarrow 0} \boldsymbol{E}[\|X - X^\epsilon\|_{\mathcal{X}_T^\kappa}^p] = 0.$$

Proof of $Z^{\epsilon} = X^{\epsilon,\unicode{x2AEF}} \to X^{\unicode{x2AEF}} = Z$

Proposition

For any $k \in \mathbf{Z}^3$ and $t \in \mathbf{R}$, $\hat{Z}_t(k)$ is a mean-zero **C***-val. Gaussian r.v. with covariance given by*

$$\boldsymbol{E}[\hat{Z}_s(k)\overline{\hat{Z}_t(l)}] = \delta_{kl} \cdot \frac{e^{-4\pi^2 \mathrm{i}|k|^2(s-t)-(4\pi^2\mu|k|^2+1)|s-t|}}{4\pi^2\mu|k|^2+1},$$

$$\boldsymbol{E}[\hat{Z}_s(k)\hat{Z}_t(l)] = 0 = \boldsymbol{E}[\hat{Z}_s(k)\hat{Z}_t(l)].$$

Proof of $Z^\epsilon = X^{\epsilon,\mathrm{I}} \to X^{\mathrm{I}} = Z$

Since $\hat{Z}_t^\epsilon = \chi^\epsilon(k)\hat{Z}_t$, we have

$$[\triangle_m(Z_t - Z_t^\epsilon)](x) = \sum_{k\in\mathbf{Z}^3} \rho_m(k)\{1-\chi^\epsilon(k)\}\hat{Z}_t(k)\mathbf{e}_k(x).$$

Let $0 < h < \kappa < 1$. We obtain

$$\begin{aligned}
\boldsymbol{E}[|[\triangle_m(Z_t - Z_t^\epsilon](x)|^2] \\
&= \sum_{k\in\mathbf{Z}^3} \rho_m(k)^2\{1-\chi^\epsilon(k)\}^2 \boldsymbol{E}[|\mathbf{e}_k(x)|^2] \\
&\lesssim \sum_{k\in\mathbf{Z}^3} \rho_m(k)^2(\epsilon|k|)^{2h}(|k|^2+1)^{-1} \\
&\lesssim \epsilon^{2h}(2^m)^{1+2h}.
\end{aligned}$$

Proof of $Z^\epsilon = X^{\epsilon,\mid} \to X^{\mid} = Z$

Hence, for $\alpha = -1/2 - \kappa$, we have

$$
\begin{aligned}
\boldsymbol{E}[\|Z_t - Z_t^\epsilon\|_{\mathcal{C}^\alpha}^{2p}] &\lesssim \sum_{m=-1}^{\infty} 2^{(2\alpha p+1)m} \{\epsilon^{2h}(2^m)^{1+2h}\}^p \\
&= \epsilon^{2hp} \sum_{m=-1}^{\infty} 2^{(1-2(\kappa-h)p)m} \\
&\to 0.
\end{aligned}
$$

Proof of convergence of $X^{\epsilon,\circ\!\vee\!\circ}$, $X^{\epsilon,\circ\!\vee\!\bullet}$,...

We calculate integrability of the kernels of the Itô-Wiener integral $X^{\epsilon,\tau}$. To this end, we use

- the product formula for the Itô-Wiener integral,
- the Fourier transform wrt time parameters of the kernels

Thank you for your attention.