

Stochastic complex Ginzburg-Landau equation with space-time white noise *

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In this talk, we prove local well-posedness of the stochastic complex Ginzburg-Landau equation with a complex-valued space-time white noise ξ in the three-dimensional torus $\mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$

$$(CGLE) \quad \begin{cases} \partial_t u = (i + \mu)\Delta u + \nu(1 - |u|^2)u + \xi & \text{on } (0, \infty) \times \mathbf{T}^3, \\ u(0, \cdot) = u_0(\cdot). \end{cases}$$

Here, $i = \sqrt{-1}$, μ is a positive constant and ν is a complex constant.

Before starting our discussion, we introduce notation. We denote by $\mathcal{D}$ the space of all smooth functions on $\mathbf{T}^3$ and by $\mathcal{D}'$ its dual. For every $\alpha \in \mathbf{R}$, $1 \leq p, q \leq \infty$, we denote by $\mathcal{B}_{p,q}^\alpha$ the Besov space, which is defined by the completion of the space of smooth functions on $\mathbf{T}^3$ under the Besov norm $\|\cdot\|_{\mathcal{B}_{p,q}^\alpha}$. To define the Besov norm, we use the Littlewood-Paley block $\{\Delta_m = \mathcal{F}^{-1}\rho_m\mathcal{F}\}_{m=-1}^\infty$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transformation and its inverse, respectively, and $\{\rho_m\}_{m=-1}^\infty$ is the dyadic partition of unity. For notational simplicity, we set the Hölder-Besov space $\mathcal{C}^\alpha = \mathcal{B}_{\infty,\infty}^\alpha$ and denote by $C_T\mathcal{C}^\alpha$ the space of all $\mathcal{C}^\alpha$ -valued continuous functions on $[0, T]$ for every $T > 0$. Next we introduce the notion of paradifferential calculus. For every $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, we define the resonance $f \odot g$ and the paraproduct $f \otimes g$. They give the decomposition $fg = f \otimes g + f \odot g + f \otimes g$. The paraproduct $f \otimes g$ can be defined for any $\alpha, \beta \in \mathbf{R}$, but the resonance $f \odot g$ can be defined for $\alpha + \beta > 0$. Hence, in order to define products fg , it is necessary that $\alpha + \beta > 0$ holds. Finally, we set $\mathcal{L}^1 = \partial_t - \{(i + \mu)\Delta - 1\}$, $P_t^1 = e^{t\{(i + \mu)\Delta - 1\}}$ and $I(u)_t = \int_{-\infty}^t P_{t-s}^1 u_s ds$ for $u : [0, \infty) \rightarrow \mathcal{D}'$.

Now we return to well-posedness of the equation (CGLE). For some reason, we write (CGLE) as $\mathcal{L}^1 u = \nu(1 - |u|^2)u + u + \xi$ and discuss the problem. To illustrate difficulty of this problem, we consider a stationary solution to the linear equation $\mathcal{L}^1 Z = \xi$ on $(0, \infty) \times \mathbf{T}^3$. The solution is given by $Z_t = I(\xi)_t$ formally and it is not a function but a distribution with respect to the space variable in the dimension three. More precisely, Z_t belongs to $\mathcal{C}^{-\frac{1}{2}-\kappa}$ for any $\kappa > 0$. Hence the products Z_t^2 , $Z_t \overline{Z_t}$, $Z_t^2 \overline{Z_t}$ and so on are not defined a priori. Since the irregularity of the solution to (CGLE) comes from the white-noise, it is natural to guess that the space regularity of u_t is not better than that of Z_t and that the product $|u_t|^2 u_t = u_t^2 \overline{u_t}$ is not defined a priori.

To overcome this difficulty, we use the theory of paracontrolled distributions developed in [GIP15]. The method consists a deterministic part and a probabilistic part.

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In the deterministic part, we construct the solution map of (CGLE) from the space $\mathcal{X}_{T_*}^{\kappa}$ of driving vectors to the space $\mathcal{D}_{T_*}^{\kappa, \kappa'}$ of solutions, where T_* is a life time of a solution and κ, κ' are positive small parameters, and show that the solution map is continuous. To be precise, for every $0 < \kappa < \kappa' < 1/18$ and $T > 0$, we call a vector of space-time distributions

$$X = (X^{\mathbb{I}}, X^{\mathbb{V}}, X^{\mathbb{V}}, X^{\mathbb{Y}}, X^{\mathbb{Y}^{\circ}}, X^{\mathbb{Y}^{\circ}}, X^{\mathbb{V}}, X^{\mathbb{V}^{\circ}}, X^{\mathbb{V}^{\circ}}, X^{\mathbb{V}^{\circ}}, X^{\mathbb{V}^{\circ}}, X^{\mathbb{V}^{\circ}}, X^{\mathbb{V}^{\circ}}, X^{\mathbb{V}^{\circ}})$$

$$\in C_T \mathcal{C}^{-\frac{1}{2}-\kappa} \times (C_T \mathcal{C}^{-1-\kappa})^2 \times (C_T \mathcal{C}^{1-\kappa})^2 \times \mathcal{L}_T^{\frac{1}{2}-\kappa, \frac{1}{4}-\frac{1}{2}\kappa} \times (C_T \mathcal{C}^{-\kappa})^6 \times (C_T \mathcal{C}^{-\frac{1}{2}-\kappa})^2$$

which satisfies $\mathcal{L}^1 X^\mathbf{Y} = X^\mathbf{V}$ and $\mathcal{L}^1 X^\mathbf{Y}^\circ = X^\mathbf{V}^\circ$ a *driving vector* of (CGLE). We denote by $\mathcal{X}_T^\kappa$ the set of all driving vector. The definition of $\mathcal{D}_{T_*}^{\kappa, \kappa'}$ is a little complicated. Because we transform (CGLE) to a system of two equations with respect to (v, w) so that $u = X^1 - \nu X^\mathbf{Y}^\circ + v + w$ solves (CGLE). The space $\mathcal{D}_{T_*}^{\kappa, \kappa'}$ is where (v, w) lives.

We explain the meanings of the graphical symbols $\mathbf{I}$, $\mathbf{V}$, $\mathbf{\bar{V}}$, $\mathbf{Y}$, $\dots$. They are just coordinates mathematically; however, the dot and the line are icons for the white noise and the operation I , respectively. Hence, $\mathbf{I}$ represents $I(\xi) = Z$. Moreover, $\mathbf{\bar{I}}$ and $\mathbf{\bar{V}}$ are icons for the complex conjugate of Z and the product $Z\bar{Z}$, respectively. So $\mathbf{\bar{V}}$ means $I(Z^2\bar{Z})$. Finally, $\mathbf{\smile}$ denotes the resonance term; $\mathbf{\bar{V}}\mathbf{\smile}$ represents $I(Z^2\bar{Z}) \odot Z$.

In the probabilistic part, we construct a driving vector X^ϵ from a smeared noise ξ^ϵ with a parameter $0 < \epsilon < 1$ and show convergence of X^ϵ as $\epsilon \downarrow 0$. Of course, we assume that $\xi^\epsilon \rightarrow \xi$ as $\epsilon \downarrow 0$. More precisely, we set $X^{\epsilon, \mathbf{I}} = Z^\epsilon = I(\xi^\epsilon)_t$, $X^{\epsilon, \mathbf{I}} = \overline{Z^\epsilon}$ and $X^{\epsilon, \mathbf{V}} = (Z^\epsilon)^2$; however, since $\mathbf{c}_1^\epsilon = \mathbf{E}[Z_t^\epsilon \overline{Z_t^\epsilon}]$ diverges as $\epsilon \downarrow 0$, we need to consider renormalization and set $X^{\epsilon, \mathbf{V}} = Z^\epsilon \overline{Z^\epsilon} - \mathbf{c}_1^\epsilon$. In order to define $X^{\epsilon, \tau}$ for $\mathbf{V}, \mathbf{V}^{\circ}, \mathbf{V}^{\circ}, \mathbf{V}^{\circ}$ and $\mathbf{V}^{\circ}$, it is necessary to consider renormalization. The other renormalization constants are $\mathbf{c}_{2,1}^\epsilon = \frac{1}{2} \mathbf{E}[X_{(t,x)}^{\epsilon, \mathbf{Y}} \odot X_{(t,x)}^{\epsilon, \mathbf{V}^{\circ}}]$ and $\mathbf{c}_{2,2}^\epsilon = \mathbf{E}[X_{(t,x)}^{\epsilon, \mathbf{Y}^\circ} \odot X_{(t,x)}^{\epsilon, \mathbf{V}}]$. To show convergence of X^ϵ , we express $\Delta_m X^\tau$ by the Itô-Wiener integrals and estimate their kernels.

From the discussion above, we obtain our main result:

Theorem 1. *Set $\mathfrak{c}^\epsilon = 2(\mathfrak{c}_1^\epsilon - \overline{\nu \mathfrak{c}_{2,1}^\epsilon} - 2\nu \mathfrak{c}_{2,2}^\epsilon)$. Let $u_0 \in \mathcal{C}^{-\frac{2}{3}+\kappa'}$. Consider the renormalized equation*

$$(CGLE') \quad \begin{cases} \partial_t u^\epsilon = (i + \mu) \Delta u^\epsilon + \nu(1 - |u^\epsilon|^2)u^\epsilon + \nu \mathfrak{C}^\epsilon u^\epsilon + \xi^\epsilon, & \text{on } (0, \infty) \times \mathbf{T}^3, \\ u(0, \cdot) = u_0(\cdot). \end{cases}$$

Then $\mathfrak{c}^\epsilon \rightarrow \infty$ as $\epsilon \downarrow 0$ and there exists a unique process u^ϵ and a random time T_*^ϵ such that

- u^ϵ solves (CGLE') on $[0, T_*^\epsilon) \times \mathbf{T}^3$,
- T_*^ϵ converges to some a.s. positive random time T_* in probability,
- u^ϵ converges to some process u defined on $[0, T_*) \times \mathbf{T}^3$ in the sense that $u^\epsilon \rightarrow u$ in probability in $C_t \mathcal{C}^{-\frac{3}{2} + \kappa'}$ for every $0 < t < T_*$. Furthermore, u is independent to the choice of ξ^ϵ .

References

- [GIP15] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.