Global well-posedness of singular stochastic PDEs

Masato Hoshino

The University of Tokyo

November 14, 2016

Introduction

- There are recently developed theories by
  - Hairer,
  - Gubinelli-Imkeller-Perkowski,
  - Kupiainen,

  which give meanings to singular stochastic PDEs
  - KPZ,
  - Φ_2^4, Φ_3^4,
  - PAM,
  - SNS,...
- General theory ignores specific properties of nonlinear terms, so (except PAM) we can obtain only local-in-time existence in general.
- In this talk, we show how to obtain global-in-time existence for the two examples (we consider both of them in the torus):
  - Coupled KPZ equations,
  - Complex stochastic Ginzburg-Landau equation.

Coupled KPZ equations

- Ferrari, Sasamoto and Spohn (2013) discussed the equation of $h = (h^\alpha)_{\alpha=1}^d$:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_\beta \xi^\beta, \quad t > 0,\ x \in \mathbb{T}, \quad \text{(KPZ)}$$

  where $(\xi^\alpha)_{\alpha=1}^d$ are independent space-time white noises, $(\sigma^\alpha_\beta)_{\alpha,\beta=1}^d$ and $(\Gamma^\alpha_{\beta\gamma})_{\alpha,\beta,\gamma=1}^d$ are given constants. σ is invertible and Γ^α is symmetric:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}.$$

- The equation (KPZ) is expected to appear as a space-time scaling limit of a microscopic system which has d conserved quantities.
- A similar equation appears in Hairer's work as a motion of loops on a d-dim manifold, although then Γ and σ are functions of u.

Ill-posedness of the KPZ euqation

- $\mathcal{C}^\alpha :=$ the completion of $C^\infty(\mathbb{T})$ under the $\mathcal{B}^\alpha_{\infty,\infty}$-norm.
- The solution $h = (h^\alpha)_{\alpha=1}^d$ of coupled KPZ equations is expected to belong to the space $C(\mathbb{R}_+, (\mathcal{C}^{\frac{1}{2}-\kappa})^d)$ a.s. for every $\kappa > 0$.
- The nonlinear term $\partial_x h^\beta \partial_x h^\gamma$ is a product of elements of $\mathcal{C}^{-\frac{1}{2}-\kappa}$, so that ill-posed.
- For a scalar valued case

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi,$$

  the Cole-Hopf transform $Z = e^h$ formally solves the multiplicative SHE

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\xi.$$

  $h = \log Z$ is the so-called Cole-Hopf solution.
- For a multi-component case, such transform does not work in general.

Local well-posedness

Theorem (Funaki-H)

Let $\xi^{\epsilon,\alpha}(t,x) = (\xi^\alpha(t) * \eta^\epsilon)(x)$ be a smeared noise by an even and smooth mollifier $\eta^\epsilon(x) = \epsilon^{-1}\eta(\epsilon^{-1}x)$. There exists a constant matrix $(B^{\epsilon,\beta\gamma})_{\beta,\gamma=1}^d = \mathcal{O}(\log \epsilon^{-1})$ and the solution $h^\epsilon = (h^{\epsilon,\alpha})_{\alpha=1}^d$ of

$$\partial_t h^{\epsilon,\alpha} = \frac{1}{2}\partial_x^2 h^{\epsilon,\alpha} + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^{\epsilon,\beta}\partial_x h^{\epsilon,\gamma} - c^\epsilon A^{\beta\gamma} - B^{\epsilon,\beta\gamma}) + \sigma^\alpha_\beta \xi^{\epsilon,\beta},$$
$$h^\epsilon(0,\cdot) = h_0,$$

where $c^\epsilon = \epsilon^{-1}\int \eta^2(x)dx$ and $A^{\beta\gamma} = \sum_\delta \sigma^\beta_\delta \sigma^\gamma_\delta$, converges locally in time to a universal limit h as $\epsilon \downarrow 0$.

- When $d = 1$, the limit h coincides with the Cole-Hopf solution.

Global existence

- Global in time existence of the limit h is derived by studying an invariant measure.
- The tilt process $u = \partial_x h$ has an invariant measure in scalar valued case.
- Funaki (2015) showed the infinitesimal invariance under the conditions

$$\sigma = I, \quad \Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\beta}_{\alpha\gamma}.$$

- The equation (KPZ) is equivalent to

$$\partial_t \hat{h}^{\alpha} = \tfrac{1}{2}\partial_x^2 \hat{h}^{\alpha} + \tfrac{1}{2}\hat{\Gamma}^{\alpha}_{\beta\gamma}\partial_x \hat{h}^{\beta}\partial_x \hat{h}^{\gamma} + \xi^{\alpha},$$

where $\hat{h}^{\alpha} = (\sigma^{-1})^{\alpha}_{\beta} h^{\beta}$ and $\hat{\Gamma}^{\alpha}_{\beta\gamma} = (\sigma^{-1})^{\alpha}_{\alpha'}\Gamma^{\alpha'}_{\beta'\gamma'}\sigma^{\beta'}_{\beta}\sigma^{\gamma'}_{\gamma}$.
- If the trilinear condition

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} = \hat{\Gamma}^{\alpha}_{\gamma\beta} = \hat{\Gamma}^{\beta}_{\alpha\gamma} \tag{TL}$$

holds, the equation (KPZ) is expected to have a Gaussian invariant measure.

Main result

- Coupled stochastic Burgers equation of $u \equiv (u^\alpha)_{\alpha=1}^d = \partial_x h$:

$$\partial_t u^\alpha = \tfrac{1}{2}\partial_x^2 u^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x(u^\beta u^\gamma) + \sigma^\alpha_\beta \partial_x \xi^\beta, \quad t>0,\ x\in\mathbb{T}, \quad \text{(SBE)}$$

on the space $(\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d = \{u \in (\mathcal{C}^{-\frac{1}{2}-\kappa})^d\,;\, \int_{\mathbb{T}} u(x)dx = 0\}$.

- Let μ_A be the distribution of a centered Gaussian noise $\eta = (\eta^\alpha(x))_{1\le\alpha\le d, x\in\mathbb{T}}$ with covariance

$$\mathbb{E}[\eta^\alpha(x)\eta^\beta(y)] = A^{\alpha\beta}\delta(x-y).$$

Theorem (Funaki-H)

If the trilinear condition (TL) *holds, then for* μ_A*-a.e. initial value* $u_0 \in (\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d$*, there exists a unique solution* u *of* (SBE) *on* $[0,\infty)$ *a.s. Moreover, the solution* u *is a Markov process on* $(\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d$ *which admits* μ_A *as an invariant measure.*

Main result

Corollary (Funaki-H)

If the trilinear condition (TL) *holds, then for* μ_A*-a.e.* $u_0 \in (\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d$*, there exists a unique solution* h *of* (KPZ) *on* $[0,\infty)$ *a.s. when* $\partial_x h_0 = u_0$*.*

- Hairer and Mattingly (2016) showed that the Markov process u is strong Feller. (In precise, the state space is taken as

$$\overline{(\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d} = (\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d \cup \{\Delta\}$$

by defining $u(t) = \Delta$ when $t \geq T_*$.) As a result, the solution h of (KPZ) exists on $[0,\infty)$ for all initial values $h_0 \in (\mathcal{C}^{\frac{1}{2}-\kappa})^d$ since μ_A has a dense support on $(\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d$.

Linearizable case

- If there exists an invertible matrix $s = (s^\alpha_\beta)^d_{\alpha,\beta=1}$ such that
$$\Gamma^\alpha_{\beta\gamma} = \sum_{\alpha'} (s^{-1})^\alpha_{\alpha'} s^{\alpha'}_\beta s^{\alpha'}_\gamma,$$
then $\tilde{h}^\alpha = s^\alpha_\beta h^\beta$ satisfies
$$\partial_t \tilde{h}^\alpha = \tfrac{1}{2}\partial_x^2 \tilde{h}^\alpha + \tfrac{1}{2}(\partial_x \tilde{h}^\alpha)^2 + s^\alpha_\beta \sigma^\beta_\gamma \xi^\gamma.$$
- In this case, the Cole-Hopf transform $Z^\alpha = e^{\tilde{h}^\alpha}$ works, so that global existence of h for all initial values is trivial.
- Since each component $\tilde{u}^\alpha = \partial_x \tilde{h}^\alpha$ has an Gaussian invariant measure μ_α, there exists an invariant measure μ of $\tilde{u} = (\tilde{u}^\alpha)^d_{\alpha=1}$, whose marginals coincide with μ_α. However, the explicit form of μ is unclear.

Proof of the main result

- $P_N = \varphi(ND)$: Fourier multiplier operator defined by an even and smooth cut-off function $\varphi \in C_0^\infty((-1,1),[0,1])$ such that $\varphi(0) = 1$.
- We consider the Galerkin approximation

$$\begin{aligned} &\partial_t u^{N,\alpha} = \tfrac{1}{2}\partial_x^2 u^{N,\alpha} + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x P_N(P_N u^{N,\beta} P_N u^{N,\gamma}) + \sigma^\alpha_\beta \partial_x \xi^\beta, \\ &u^N(0,\cdot) = u_0 \in (\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d. \end{aligned} \tag{GA}$$

- $u^N \to u$: unique solution of (SBE) until the time $T_* > 0$.
- μ_A is invariant under the OU process $\partial_t u = \frac{1}{2}\partial_x^2 u + \sigma \partial_x \xi$.
- μ_A is invariant under (GA), since

$$(A^{-1})_{\alpha\alpha'}\langle \Gamma^\alpha_{\beta\gamma}\partial_x P_N(P_N u^{N,\beta} P_N u^{N,\gamma}), u^{N,\alpha'}\rangle_{L^2(\mathbb{T})} = 0.$$

Proof of the main result

- (GA) has a global solution and satisfies

$$\sup_N \int_{(\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d} \mathbb{E}[\sup_{t\in[0,T]} \|u^N(t)\|^p_{(\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d}]\mu_A(du_0) < \infty,$$

  for every $T > 0$ and $p > 1$.
- $L_N := \sup_{t\in[0,T]} \|u^N(t)\|_{(\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d}$ is bounded in $L^p(\mathbb{P} \times \mu_A)$, so that $\exists\{N_k\}$ subsequence such that

$$L_{N_k} \to \exists L \quad \text{weak*}.$$

- $u^{N_k} \to u$: sol of (SBE) on $[0, T_*)$. Therefore

$$\sup_{t\in[0,T_*\wedge T)} \|u(t)\|_{(\mathcal{C}_0^{-\frac{1}{2}-\kappa})^d} \leq L < \infty, \quad \mathbb{P} \times \mu_A\text{-a.s.}$$

- $u(t)$ does not explode a.s. for μ_A-a.e. u_0.

Complex stochastic Ginzburg-Landau equation

- We study the complex-valued equation

$$\partial_t u = (i+\mu)\Delta u - \nu |u|^2 u + \lambda u + \xi, \quad t > 0,\ x \in \mathbb{T}^3. \qquad \text{(CGL)}$$

- ξ is a complex-valued space-time white noise.
- $\mu > 0$ and $\nu, \lambda \in \mathbb{C}$.
- (CGL) describes the amplitude of the slow modulation in space and time near the threshold for an instability.
- $\mu \downarrow 0 \Rightarrow$ Nonlinear stochastic Schrödinger equation?

Paracontrolled ansatz

- Set $u = \underbrace{Z}_{-1/2-} - \nu \underbrace{W}_{1/2-} + \underbrace{\tilde{u}}_{1-}$, where

$$\begin{aligned}
\partial_t Z &= (i+\mu)\Delta Z + \xi, \\
\partial_t W &= (i+\mu)\Delta W + Z^2 \bar{Z}, \\
\partial_t \tilde{u} &= (i+\mu)\Delta \tilde{u} - \nu(\tilde{u} - \nu W)^2(\bar{u} - \bar{\nu}\bar{W}) \\
&\quad - \nu\{(\tilde{u} - \nu W)^2 \bar{Z} + 2(\tilde{u} - \nu W)(\bar{\tilde{u}} - \bar{\nu}\bar{W})\} \\
&\quad - \nu\{2 \underbrace{(\tilde{u} - \nu W)}_{1/2-} \underbrace{Z\bar{Z}}_{-1-} + \underbrace{(\bar{\tilde{u}} - \bar{\nu}\bar{W})}_{1/2-} \underbrace{Z^2}_{-1-}\} + \text{others.}
\end{aligned}$$

Paracontrolled ansatz

- Set $\tilde{u} = \underbrace{v}_{1-} + \underbrace{w}_{3/2-}$, where

$$\partial_t v = (i+\mu)\Delta v - \nu\{2(v+w-\nu W) \otimes Z\bar{Z} + (\bar{v}+\bar{w}-\bar{\nu}\bar{W}) \otimes Z^2\}$$
$$=: (i+\mu)\Delta v + F(v,w),$$
$$\partial_t w = (i+\mu)\Delta w - \nu\{2(v+w-\nu W) \odot Z\bar{Z} + (\bar{v}+\bar{w}-\bar{\nu}\bar{W}) \odot Z^2\}$$
$$+ \text{others} =: (i+\mu)\Delta w + G(v,w).$$

- v has the form

$$v = -\nu\{2(v+w-\nu W) \otimes \underbrace{Y_1}_{1-} + (\bar{v}+\bar{w}-\bar{\nu}\bar{W}) \otimes \underbrace{Y_2}_{1-}\} + (\mathcal{C}^{3/2-}),$$

where

$$\partial_t Y_1 = (i+\mu)\Delta Y_1 + Z\bar{Z}, \quad \partial_t Y_2 = (i+\mu)\Delta Y_2 + Z^2.$$

Therefore $v \odot Z\bar{Z}$ and $\bar{v} \odot Z^2$ are defined if $Y_1 \odot Z\bar{Z}$, $Y_2 \odot Z\bar{Z}$, $\bar{Y}_1 \odot Z^2$, and $\bar{Y}_2 \odot Z^2$ are given.

Local well-posedness

- $0 < \kappa < \kappa' < 1/18$: small, $c > 0$: large.
- Drivers

$$\mathbf{Z} = (\underset{-1/2-\kappa}{Z}, \underset{1/2-\kappa}{W}, W \underset{-\kappa}{\odot} Z, W \underset{-\kappa}{\odot} \bar{Z}, \underset{-1-\kappa}{Z^2}, \underset{-1-\kappa}{Z\bar{Z}}, \underset{1-\kappa}{Y_1}, \underset{1-\kappa}{Y_2},$$
$$Y_1 \underset{-\kappa}{\odot} Z\bar{Z}, Y_2 \underset{-\kappa}{\odot} Z\bar{Z}, \bar{Y}_1 \underset{-\kappa}{\odot} Z^2, \bar{Y}_2 \underset{-\kappa}{\odot} Z^2, W \underset{-1/2-\kappa}{\odot} Z\bar{Z}, \bar{W} \underset{-1/2-\kappa}{\odot} Z^2).$$

- We consider the system

$$\begin{cases} \partial_t v = (i+\mu)\Delta v + F(v,w) - cv, \\ \partial_t w = (i+\mu)\Delta w + G(v,w) + cv. \end{cases} \tag{CGL'}$$

Proposition (H-Inahama-Naganuma)

For every sequence $\mathbf{Z}$ *and initial condition* $(v_0, w_0) \in \mathcal{C}^{-\frac{2}{3}+\kappa'} \times \mathcal{C}^{-\frac{1}{2}-2\kappa}$, *the system* (CGL') *has a unique solution locally in time.*

Local well-posedness

Theorem (H-Inahama-Naganuma)

Let $\xi^\epsilon(t,x) = (\xi(t) * \eta^\epsilon)(x)$ be a smeared noise by a smooth mollifier $\eta^\epsilon(x) = \epsilon^{-1}\eta(\epsilon^{-1}x)$. There exists a constant $C^\epsilon = \mathcal{O}(\epsilon^{-1})$ and the solution u^ϵ of

$$\partial_t u^\epsilon = (i+\mu)\Delta u^\epsilon - \nu|u^\epsilon|^2 u^\epsilon + C^\epsilon u^\epsilon + \xi^\epsilon,$$
$$u^\epsilon(0,\cdot) = u_0$$

converges locally in time to a universal limit u as $\epsilon \downarrow 0$.

Global well-posedness of Φ^4_3

- Mourrat and Weber showed global well-posedness of the real-valued equation

$$\partial_t u = \Delta u - u^3 + mu + \xi, \quad t > 0,\ x \in \mathbb{T}^3,$$

where $m \in \mathbb{R}$.

- Our proof is entirely based on theirs, but partly improves it.

Main result

Theorem (H)

Let $\mu > \frac{1}{2\sqrt{2}}$ and $\Re\nu > 0$. Let $0 < \kappa < \kappa'$ be small depending on μ. For every $T > 0$ and $\mathbf{Z}$, there exists large $c > 0$ such that the following result holds. For every $(v_0, w_0) \in \mathcal{C}^{-\frac{2}{3}+\kappa'} \times \mathcal{C}^{-\frac{1}{2}-2\kappa}$, there exists $C > 0$ such that any solution (v, w) of the system (CGL') on $[0, T]$ satisfies

$$\sup_{t\in[0,T]} \left(\|v(t)\|_{\mathcal{C}^{-\frac{2}{3}+\kappa'}} + \|w(t)\|_{\mathcal{C}^{-\frac{1}{2}-2\kappa}}\right) \leq C.$$

- Although the system (CGL') depends on $c > 0$, the required process $u = Z - \nu W + v + w$ does not.

Space $\mathcal{B}_p^\alpha$

- Instead of $\mathcal{C}^\alpha$, we use $\mathcal{B}_p^\alpha$ ($p \in [1, \infty)$) in the proof.
- $u = \sum_i \Delta_i u$ (Littlewood-Paley decomposition)

$$\|u\|_{\mathcal{B}_p^\alpha} = \sup_i 2^{\alpha i} \|\Delta_i u\|_{L^p}.$$

Proposition

Let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

1. $\|u \varolessthan v\|_{\mathcal{B}_r^\beta} \lesssim \|u\|_{L^p} \|v\|_{\mathcal{B}_q^\beta}$.
2. $\alpha < 0 \Rightarrow \|u \varolessthan v\|_{\mathcal{B}_r^{\alpha+\beta}} \lesssim \|u\|_{\mathcal{B}_p^\alpha} \|v\|_{\mathcal{B}_q^\beta}$.
3. $\alpha + \beta > 0 \Rightarrow \|u \odot v\|_{\mathcal{B}_r^{\alpha+\beta}} \lesssim \|u\|_{\mathcal{B}_p^\alpha} \|v\|_{\mathcal{B}_q^\beta}$.

In particular, $\alpha + \beta > 0 \Rightarrow \|uv\|_{\mathcal{B}_r^{\alpha \wedge \beta}} \lesssim \|u\|_{\mathcal{B}_p^\alpha} \|v\|_{\mathcal{B}_q^\beta}$.

Strategy

Let $p > 1$ and $q = \frac{2p+2}{3}$.

1. Control $\|v(t)\|^{2p+2}_{\mathcal{B}^{\frac{1}{2}+\kappa'}_{2p+2}}$ by $\|w(t)\|^{2p+2}_{L^{2p+2}}$ (Gronwall type inequality).
2. Controll $\|w(t)\|^{2p+2}_{L^{2p+2}}$ by $\|w(t)\|^{q}_{\mathcal{B}^{1+2\kappa'}_q}$ (L^{2p} inequality).
3. Controll $\|w(t)\|^{q}_{\mathcal{B}^{1+2\kappa'}_q}$ by
$$(\text{small}) \times \underbrace{(\|v(t)\|^{2p+2}_{\mathcal{B}^{\frac{1}{2}+\kappa'}_{2p+2}} + \|w(t)\|^{2p+2}_{L^{2p+2}} + \|w(t)\|^{q}_{\mathcal{B}^{1+2\kappa'}_q})}_{\Phi(t)}.$$
4. A priori $L^1[0,T]$ estimate of $\Phi(t)$.
5. Step 4 $\Rightarrow$ $L^\infty[0,T]$ estimate of $\|v(t)\|_{\mathcal{B}^{\frac{1}{2}+\kappa'}_{2p+2}}$.
6. Step 4, $p > \frac{3}{2}$ $\Rightarrow$ $L^\infty[0,T]$ estimate of $\|w(t)\|_{\mathcal{B}^{\frac{3}{2}-2\kappa'}_q}$.

Step1

$$\partial_t v = (i+\mu)\Delta v - \nu\{2(v + w - \nu W) \otimes Z\bar{Z} + (\bar{v} + \bar{w} - \bar{\nu}\bar{W}) \otimes Z^2\} - cv.$$

Proposition

For every $p \geq 1$ and $t \in [0, T]$,

$$\|v_t\|_{L^{2p+2}} \lesssim e^{-ct}\|v_0\|_{L^{2p+2}} + \int_0^t e^{-c(t-s)}(t-s)^{-\frac{1+\kappa'}{2}}(1 + \|w_s\|_{L^{2p+2}})ds,$$

$$\|v_t\|_{\mathcal{B}_{2p+2}^{\frac{1}{2}+\kappa'}} \lesssim \|v_0\|_{\mathcal{B}_{2p+2}^{\frac{1}{2}+\kappa'}} + \int_0^t (t-s)^{-\frac{3}{4}-\kappa'}(1 + \|w_s\|_{L^{2p+2}})ds.$$

Step2

- $\partial_t w = (i+\mu)\Delta w - \nu w^2 \bar{w} + \cdots$
- Doering-Gibbon-Levermore (1994) showed L^{2p}-inequality for the solution of deterministic CGL when

$$1 < p < 1 + \mu(\mu + \sqrt{1+\mu^2}).$$

Proposition

Choose sufficiently large $c > 0$. *For every* $p \in (1, 5 \wedge \{1 + \mu(\mu + \sqrt{1+\mu^2})\})$ *and* $t \in [0, T]$,

$$\|w_t\|_{L^{2p}}^{2p} + \int_0^t \|w_s\|_{L^{2p+2}}^{2p+2} ds$$

$$\lesssim 1 + \|v_0\|_{\mathcal{B}_{2p+2}^{\frac{1}{2}+\kappa'}}^{2p+2} + \|w_0\|_{L^{2p}}^{2p} + \int_0^t \|w_s\|_{\mathcal{B}_q^{1+2\kappa'}}^q ds.$$

Proposition

There exists $T_* > 0$ *(indep. of* (v_0, w_0)*) such that for every* $s < t \in [0, T]$ *with* $t - s \le 2T_*$*,*

$$\int_s^t \|w_r\|^q_{\mathcal{B}_q^{1+2\kappa'}} dr \lesssim 1 + \|v_s\|^{2p+2}_{\mathcal{B}_{2p+2}^{\frac{1}{2}+\kappa'}} + \|w_s\|^{2p+2}_{L^{2p+2}} + \|w_s\|^q_{\mathcal{B}_q^{1+2\kappa'}} = 1 + \Phi(s).$$

Key estimate: $\|w^2\|^q_{\mathcal{B}_q^{\frac{1}{2}+\kappa'}} \lesssim \|w\|^{2p+2}_{L^{2p+2}} + \|w\|^q_{\mathcal{B}_q^{1+2\kappa'}}$. ($\Leftarrow$ Bony's decomposition and interpolation.)

Corollary

Under the assumptions above,

$$\int_0^T \Phi(t)dt \le C = C(v_0, w_0, T) < \infty.$$

Step5

Recall the estimate in Step1:

$$\|v_t\|_{\mathcal{B}^{\frac{1}{2}+\kappa'}_{2p+2}} \lesssim \|v_0\|_{\mathcal{B}^{\frac{1}{2}+\kappa'}_{2p+2}} + \int_0^t (t-s)^{-\frac{3}{4}-\kappa'}(1+\|w_s\|_{L^{2p+2}})ds.$$

Proposition

$$\sup_{t\in[0,T]} \|v_t\|_{\mathcal{B}^{\frac{1}{2}+\kappa'}_{2p+2}} \lesssim 1.$$

Step6

- $w_t = e^{t(i+\mu)\Delta} w_0 + \sum_j \int_0^t e^{(t-s)(i+\mu)\Delta} G_j(s) ds.$
- Young's inequality

$$\|w\|_{L_T^p \mathcal{B}_q^{\frac{3}{2}-2\kappa'}} \lesssim 1 + \sum_j \left(\int_0^T (T-t)^{-\frac{3-4\kappa'-2\alpha_j}{4} q_j} dt \right)^{\frac{1}{q_j}} \|G_j\|_{L_T^{p_j} \mathcal{B}_q^{\alpha_j}}$$

$(1 + \frac{1}{p} = \frac{1}{q_j} + \frac{1}{p_j})$ implies the improvement of temporal integrability.

- If $p > \frac{3}{2} (\Leftrightarrow \mu > \frac{1}{2\sqrt{2}})$, then we can perform $\mathcal{O}(|\frac{\log \kappa'}{\kappa'}|)$-times improvements and obtain

Proposition

$$\sup_{t \in [0,T]} \|w_t\|_{\mathcal{B}_q^{\frac{3}{2}-2\kappa'}} \lesssim 1.$$