GLOBAL WELL-POSEDNESS OF SINGULAR STOCHASTIC PDES

MASATO HOSHINO (THE UNIVERSITY OF TOKYO)

We discuss global-in-time existence of the solution of semilinear stochastic PDE of the type

$$\mathcal{L}u = F(u, \nabla u) + \xi, \quad t > 0,\ x \in \mathbb{T}^d,$$

where $\mathcal{L}$ is a parabolic operator, F is a nonlinear operator, and ξ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^d$. The main difficulty of this equation is that the nonlinear operator F is not well defined in general because we expect that $u(t, \cdot) \in \mathcal{C}^{\frac{2-d}{2}-}$. Recently, Gubinelli-Imkeller-Perkowski introduced the *paracontrolled calculus* as a tool of giving a meaning to this equation under some assumptions. They solved some singular stochastic PDEs locally in time in the following sense. We replace ξ by a smooth noise ξ^ϵ which approximate ξ in $\epsilon \downarrow 0$ and consider the solution u^ϵ of

$$\mathcal{L}u^\epsilon = M^\epsilon F(u^\epsilon, \nabla u^\epsilon) + \xi^\epsilon,$$

where M^ϵ is a suitable *renormalization* of F. Then u^ϵ converges to a universal limit u in a short time. However, global-in-time existence is not known in general. We discuss this problem for the following two examples.

First example is the coupled KPZ equation

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \xi^\alpha, \quad t > 0,\ x \in \mathbb{T}$$

for an $\mathbb{R}^d$-valued process $h = (h^\alpha)_{\alpha=1}^d$. Here $(\Gamma^\alpha_{\beta\gamma})_{1\le\alpha,\beta,\gamma\le d}$ are given constants and $\xi = (\xi^\alpha)$ is an $\mathbb{R}^d$-valued space-time white noise. This is a joint work with Tadahisa Funaki (The University of Tokyo).

Theorem 1. *Let $\xi^\epsilon(t,x) = (\xi(t) * \rho^\epsilon)(x)$ be a smeared noise with an even mollifier $\rho^\epsilon(x) = \epsilon^{-1}\rho(\epsilon^{-1}x)$. Then there exist constants $C^{\epsilon,\beta\gamma} = O(\epsilon^{-1})$ such that the solution h^ϵ of*

$$\partial_t h^{\epsilon,\alpha} = \tfrac{1}{2}\partial_x^2 h^{\epsilon,\alpha} + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^{\epsilon,\beta}\partial_x h^{\epsilon,\gamma} - C^{\epsilon,\beta\gamma}) + \xi^{\epsilon,\alpha}$$

converges to a universal limit h in a short time.

Furthermore, if we assume that

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\beta_{\alpha\gamma}$$

for all α, β, γ, then there exists a μ-full set H such that the limit h starting at h_0 with $\partial_x h_0 \in H$ exists globally in time. Here μ is the distribution of an $\mathbb{R}^d$-valued spatial white noise on $\mathbb{T}$.

Second example is the complex Ginzburg-Landau equation

$$\partial_t u = (i+\mu)\Delta u + \nu(1-|u|^2)u + \xi, \quad t>0,\ x\in\mathbb{T}^3$$

for a complex-valued process u. Here $\mu > 0$, $\nu \in \mathbb{C}$, and ξ is a complex space-time white noise. This is a joint work with Yuzuru Inahama (Kyushu University) and Nobuaki Naganuma (Osaka University).

Theorem 2. *Let $\xi^\epsilon(t,x) = (\xi(t) * \rho^\epsilon)(x)$ be a smeared noise with a mollifier $\rho^\epsilon(x) = \epsilon^{-3}\rho(\epsilon^{-1}x)$. Then there exists a constant $C^\epsilon = O(\epsilon^{-1})$ such that the solution u^ϵ of*

$$\partial_t u^\epsilon = (i+\mu)\Delta u^\epsilon + \nu(1-|u^\epsilon|^2 + C^\epsilon)u^\epsilon + \xi^\epsilon$$

converges to a universal limit u in a short time

Furthermore, if $\mu > \frac{1}{2\sqrt{2}}$ and $\Re\nu > 0$, then the limit u exists globally in time for all initial values $u_0 \in \mathcal{C}^{-2/3+}$.