Gubinelli-Imkeller-Perkowski's approach to the 3D dynamic Φ^4 model

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9 Nov 2016

3D dynamic Φ^4 model (stochastic quantization eq.)

We study the following stochastic PDE on $(0, \infty) \times \mathbf{T}^3$ for $\mathbf{R}$-valued $u = u(t, x)$:

$$\partial_t u = \triangle_x u - u^3 + \xi, \qquad \text{with } u(0, \cdot\,) = u_0.$$

Here, $\xi = \xi(t, x)$ is space-time WN on $\mathbf{R} \times \mathbf{T}^3$.
We consider *mild solutions* only. Hence, this actually solves

$$u_t = e^{t\triangle} u_0 - I(u^3)_t + X_t,$$

where $I(z)_t := \int_0^t e^{(t-s)\triangle} z_s ds$ and $X = I(\xi)$ is OU process.

But, this SPDE is highly ill-posed. The reason is as follows:

- When $u_0 \equiv 0$ and the nonlinear term $-u^3$ is absent, the solution is X itself. Its regularity for a fixed t is $(-1/2)^- := -1/2 - \delta \ (\forall \delta > 0)$.
- A natural guess: Regularity of u cannot be better than that of X, $(-1/2)^-$ at best.
  $\Longrightarrow$ u_t is just a distribution, not a function.
  $\Longrightarrow$ u_t^3 is ill-defined.

There exist 3 methods. All of them are quite new.

- Hairer's theory of regularity structures.
  Fields medal 2014 Hoshino's talk.
- Gubinelli-Imkeller-Perkowski's paracontrolled calculus, also known as theory of paracontrolled disributions.
  Catellier-Chouk's preprint. This talk.
- Kupiainen's theory based on renormalization group theory.
  Perhaps, almost nobody in Japan knows this.

Both Hairer's and GIP's theory are descendants of Gubinelli's version of rough path theory (controlled path theory).

Gubinelli's version of rough path theory

Rough path theory was invented by T. Lyons in 1998.
Now there are some variants:

- Lyons' original rough path theory,
- Gubinelli's controlled path theory,
- Lyons-Yang's new theory, which has no name yet.

The singular SPDE theories we discuss here emerged from Gubinelli's version, so one must first recall it.

Driven ODE = Controlled ODE

$x : [0,1] \to \mathbf{R}^d$, conti., "nice enough"

$\sigma : \mathbf{R}^n \to \mathrm{Mat}(n,d)$, coefficients, "nice enough"

♠ ODE controlled by x.

$$dy_t = \sigma(y_t) dx_t, \qquad y_0 \in \mathbf{R}^n.$$

Its proper definition is the following integral eq.

$$y_t = y_0 + \int_0^t \sigma(y_s) dx_s.$$

If a line integral along x on RHS can be defined, then this equation can be formulated.

♠ Sufficient conditions for line integral to make sense

- x is piecewise C^1 $\implies$ $dx_t = x'_t dt$
- x is of bdd var., or Lipschitz conti.
- x is of α-, y (or $\sigma(y)$) is of β-Hölder conti, $\alpha + \beta > 1$.
  $\implies \exists$ Young (= generalized RS) integral.

We usually use Young integral with $\alpha = \beta$. So, $\alpha = 1/2$ is a threshold. No Young integration if $0 < \alpha \leq 1/2$.

But, Brownian sample paths are $(1/2)^-$-Hölder conti.
So, deterministic line integral along Brownian paths were impossible (before the advent of RP theory).

♠ Lyons idea: Add more information to x in a systematic way and generalize the notion of paths. (=Lifting a path).

More precisely,

Not just x itself, but iterated integrals of x are also taken into consideration.
Structure of tensor algebra/nilpotent Lie group is used.

- Line integrals and driven ODEs can be defined deterministically. Itô map is continuous in this world (Lyons' continuity theorem)
- If you put rough path lift of BM into Lyons-Itô map, then you get a sol. of Stratonovich SDE.
- Loosely speaking, "de-randomization of SDE" or "separation of measure and differential equation"
- What is NOT used? martingale, Markov property, filtration. Strong taste of "real analysis"

Definition of rough path $\triangle := \{(s,t) \mid 0 \le s \le t \le 1\}$.

$1/3 < \alpha \le 1/2$ A continuous map

$X = (1, X^1, X^2) : \triangle \to \mathbf{R} \oplus \mathbf{R}^d \oplus (\mathbf{R}^d \otimes \mathbf{R}^d)$

is said to be a rough path if

(i) K. T. Chen's identity $0 \le s \le u \le t \le 1$,

$$\begin{aligned} X^1_{s,t} &= X^1_{s,u} + X^1_{u,t}, \\ X^2_{s,t} &= X^2_{s,u} + X^2_{u,t} + X^1_{s,u} \otimes X^1_{u,t}. \end{aligned}$$

(ii) α-Hölder condition $\|X^1\|_\alpha < \infty$, $\|X^2\|_{2\alpha} < \infty$, where

$$\|A\|_\alpha := \sup_{0 \le s < t \le 1} |A_{s,t}|/|t-s|^\alpha.$$

Example (Lift of a usual path)
$x : [0,1] \to \mathbf{R}^d$, Lipschitz conti., $x_0 = 0$. Set

$$X^1_{s,t} := x_t - x_s, \quad X^2_{s,t} := \int_s^t (x_u - x_s) \otimes dx_u$$

- This $X = (X^1, X^2)$ is clearly a RP. Called a lift of x.
- The geometric RP space $G\Omega_\alpha(\mathbf{R}^d)$ is defined to be the closure of this kind of lifts of the usual paths.

RP integral à la Gubinelli

- Want to line integral along $X = (X^1, X^2) \in G\Omega_\alpha(\mathbf{R}^d)$.
- Since $\alpha \leq 1/2$, existing line integral theories fail.
- He defined a Banach space $\mathcal{Q}_X^\alpha(\mathbf{R}^n)$ of integrands w.r.t. X, which consists of paths which locally behaves like X.
- This space is different for different RP X.
- Elements of $\mathcal{Q}_X^\alpha(\mathbf{R}^n)$ are called controlled paths w.r.t. X.
- Integration map along X is from a space $\mathcal{Q}_X^\alpha(\mathbf{R}^d)$ of controlled paths to another space $\mathcal{Q}_X^\alpha(\mathbf{R}^n)$ of controlled paths.

Definition of a controlled path

A triple (Y, Y', R^Y) is called an integrand ($=$ a controlled path) of $X \in G\Omega_\alpha(\mathbf{R}^d)$ if

(i) $Y \in C^\alpha([0,1], \mathbf{R}^n)$

(ii) $Y' \in C^\alpha([0,1], \mathcal{L}(\mathbf{R}^d, \mathbf{R}^n))$

(iii) $R^Y \in C^{2\alpha}(\triangle, \mathbf{R}^n)$

(iv) $Y_t - Y_s = Y'_s \cdot X^1_{s,t} + R^Y_{s,t} \qquad (0 \le s \le t \le 1)$

The last item means "behaviour of Y is similar to (or better than) that of X".

[Notation] $(Y, Y') \in$ or $(Y, Y', R^Y) \in \mathcal{Q}^\alpha_X(\mathbf{R}^n)$.

$\mathcal{Q}^\alpha_X(\mathbf{R}^n)$ is a Banach space with $\|Y'\|_\alpha + \|R^Y\|_{2\alpha} + |Y_0| + |Y'_0|$

- If $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n, d)$ (nice) and $(Y, Y') \in \mathcal{Q}_X^\alpha(\mathbf{R}^n)$,
then $\int_0^\cdot \sigma(Y_s) dX_s \in \mathcal{Q}_X^\alpha(\mathbf{R}^n)$ with "derivative" $\sigma(Y_t)$.

-Therefore, for each RP X, a solution of controlled ODE

$$Y_t = y_0 + \int_0^t \sigma(Y_s) dX_s$$

is a fixed point of the integration map on RHS in $\mathcal{Q}_X^\alpha(\mathbf{R}^n)$.

-Thus, controlled ODE are generalized for RPs.
-Lyons-Itô map $(X, y_0) \to Y$ is continuous.

Lifting Brownian motion (the only probabilistic part)

- Let μ be Wiener measure on $C_0([0,1], \mathbf{R}^d)$.
- Want to lift μ to get a measure on $G\Omega_\alpha(\mathbf{R}^d)$.
- But, a generic conti. path does not admit a deterministic lift.
- So, we use piecewise linear approximation.
- For $w \in C_0([0,1], \mathbf{R}^d)$ and $m \in \mathbf{N}$, let $w(m)$ be the dyadic piecewise linear approx. associated to $\{0, 1/2^m, 2/2^m, \ldots, 2^m/2^m = 1\}$.

Since $w(m)$ is Lipscitz, RP lift $W(m)$ exists.

$$\exists W := \lim_{m\to\infty} W(m) \qquad \mu\text{-a.s. in } G\Omega_\alpha(\mathbf{R}^d)$$

(Brownian RP or a canonical lift of BM)

If W is used as a driver X of RDE above, then sol. Y coincides a.s. with sol. of

$$dy_t = \sigma(y_t) \circ dw_t \quad \text{with given } y_0$$

Proof: Lyons' continuity theorem and Wong-Zakai's approximation theorem.

Solution of an SDE is obtained as a image of continous map **!**

Summary: Key ingredients of controlled path theory

1. First we have Wiener measure (=BM) defined on the usual path space.
2. Paths in the usual sense are given additional information in a deterministic way (= lift or enhancement).
[the deterministic part 1].
3. For each lifted object (i.e., RP), Banach spaces of controlled paths are defiend so that the integral equation under consideration makes sense. A solutions is a fixed point in such a Banach spaces. [the deterministic part 2].
4. Brownian motion admits a lift a.s. [the probabilistic part].

Paracontrolled calculus

A quick summary of Bony's paraproduct (on torus $\mathbf{T}^3$)

- $\{\rho_j \in C^\infty(\mathbf{R}^3, [0,\infty)) | j \geq -1\}$ Dyadic partition of unity. i.e., radial, $\sum_{j\geq 1} \rho_j \equiv 1$, $\rho_j(\xi) = 0$ if $|\xi| \notin [2^{j-1}, 2^{j+1}]$.
- Define $\triangle_j := \mathcal{F}^{-1}\rho_j\mathcal{F} \quad (j \geq -1)$ Littlewood-Paley block
- Using these, Besov space $\mathcal{B}^\alpha_{\infty,\infty}(\mathbf{T}^3) =: \mathcal{C}^\alpha$ is defined.

$$\|\phi\|_{\mathcal{B}^\alpha_{\infty,\infty}(\mathbf{T}^3)} := \sup_{j\geq -1}\left\{2^{\alpha j}\|\triangle_j\phi\|_{L^\infty(\mathbf{T}^3)}\right\} \qquad (\alpha \in \mathbf{R})$$

When $\alpha > 0$, it coincides with the usual Hölder space.

- For $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$, the product fg can be defined if and only if $\alpha + \beta > 0$. If so, $fg \in \mathcal{C}^{\alpha \wedge \beta}$.

- Let $f, g : \mathbf{T}^3 \to \mathbf{R}$. Decompose fg as follows:

$$fg = f \lhd g + f \circ g + f \rhd g$$

where $f \lhd g := \sum_{i<j-1} \triangle_i f \triangle_j g$, $\quad f \circ g := \sum_{|i-j| \le 1} \triangle_i f \triangle_j g$.

• Paraproduct $f \lhd g$ is always defined, but its regularity may not be so nice. Its behaviour is similar to g (if f is a function).

Let $f \in \mathcal{C}^{\alpha}$, $g \in \mathcal{C}^{\beta}$. Then,
$f \lhd g \in \mathcal{C}^{\beta}$ (if $\alpha > 0$) and $f \lhd g \in \mathcal{C}^{\alpha+\beta}$ (if $\alpha < 0$).

• Resonant term $f \circ g$ can be defined iff fg can be defined ($\iff \alpha + \beta > 0$). In that case, its regularity is nice:
$f \circ g \in \mathcal{C}^{\alpha+\beta}$ (if $\alpha + \beta > 0$)

The OU process $X = I(\xi)$ plays the role of BM.
Its sample paths belongs to

$$C_T\mathcal{C}^{-1/2-\kappa} := C([0, T], \mathcal{C}^{-1/2-\kappa}) \qquad (\forall\kappa > 0)$$

Write a generic element of this space by $\mathcal{X}$ (deterministic).

What king of quantities are needed for Φ^4_3-model to make sense? A natural answer is

$$\underset{(-1)^-}{\mathcal{X}^2}, \underset{1^-}{\mathcal{I}(\mathcal{X}^2)}, \quad \underset{(1/2)^-}{\mathcal{I}(\mathcal{X}^3)}, \underset{0^-}{\mathcal{I}(\mathcal{X}^3) \circ \mathcal{X}}, \quad \mathcal{I}(\mathcal{X}^2) \circ \mathcal{X}^2, \underset{0^-}{\mathcal{I}(\mathcal{X}^3)} \circ \underset{(-1/2)^-}{\mathcal{X}^2},$$

Note that $\vec{\mathcal{X}} = (\mathcal{X}, \mathcal{X}^2, \mathcal{I}(\mathcal{X}^2), \ldots)$ just stands for a generic element of

$$C_T\mathcal{C}^{(-1/2)^-} \times C_T\mathcal{C}^{1^-} \times C_T\mathcal{C}^{(1/2)^-} \times \cdots \times C_T\mathcal{C}^{(-1/2)^-}.$$

So, $\mathcal{X}^2$ does not necessarily mean the square of $\mathcal{X}$.
But, one true constraint is imposed:
$\mathcal{I}(\mathcal{X}^2)_t = e^{t\triangle}\mathcal{I}(\mathcal{X}^2)_0 + I(\mathcal{X}^2)_t$.
(This is something like Chen's identity for RPs.)
An element of the above product space is called a driver if it satisfies the above constraint. It plays the role of a RP.
Set $\mathcal{Z} := \{\text{drivers}\}$.

Paracontrolled distributions

Let a driver $\vec{\mathcal{X}}$ given. $u \in C_T\mathcal{C}^{(-1/2)^-}$ of the form

$$u = \mathcal{X} - \mathcal{I}(\mathcal{X}^3) + u^\dagger \lhd \mathcal{I}(\mathcal{X}^2) + u^\sharp$$

for some $u^\dagger \in C_T\mathcal{C}^{(1/2)^-}$ and $u^\sharp \in C_T\mathcal{C}^{(3/2)^-}$ is called a paracontrolled distribution w.r.t. $\mathcal{X}$.
(Something like a controlled path in RP theory)

The space of PCD is denoted by $\mathcal{D}_{\vec{\mathcal{X}}}$. It is naturally identified (?) with $C_T\mathcal{C}^{(1/2)^-} \times C_T\mathcal{C}^{(3/2)^-}$ and hence is a Banach space.

[Key] If we plug a PCD u into the RHS of $u = -I(u^3) + X$, then we have a PCD again. Thus, RHS of Φ^4_3-equation is a continuous map from $\mathcal{D}_{\vec{\mathcal{X}}}$ to itself. Its fixed point is a solution of Φ^4_3-equation in a generalized sense.

To prove this, we need deep results from Bony's paraproduct theory.

By a standard method, we can prove a unique time-local solution exists for each $\vec{\mathcal{X}}$. Moreover, $(\vec{\mathcal{X}}, u_0) \mapsto u$ is continuous.
(Analogous to Lyons' continuity theorem)

Heuristics: Why integration map is from $\mathcal{D}_{\vec{\mathcal{X}}}$ to itself?

Assume $X = \mathcal{X}$ is nice. For simplicity, set
$\Psi := u - X = -I(X^3) + u^\dagger \lhd I(X^2) + u^\sharp \in C_T\mathcal{C}^{(1/2)^-}$. Then,

$$\Gamma(u) := X - I(u^3) = X - I(X^3) - I(\Psi^3) - 3I(\Psi X^2) - 3I(\Psi^2 X).$$

Clearly, $\Psi^3 \in C_T\mathcal{C}^{(1/2)^-} \Longrightarrow I(\Psi^3) \in C_T\mathcal{C}^{(5/2)^-}$ ($\Gamma(u)^\sharp$-part).

But, ΨX^2 and $\Psi^2 X$ may be ill-defined.
We will illustrate how these terms are taken care of.

[Note] I is always well-defined. It improves the Besov regularity by 2^-.

Let's look at $I(\Psi X^2) = I(\Psi \lhd X^2) + I(\Psi \circ X^2) + I(\Psi \rhd X^2)$.
We will check RHS $= \Psi \lhd I(X^2) +$ [terms of reg. $(3/2)^-$].

- Clearly, $I(\Psi \rhd X^2)$ is of reg. $(3/2)^-$.
- By a non-trivial relation of paraproduct and I,

$$I(\Psi \lhd X^2) = \Psi \lhd I(X^2) + [\text{a term of reg. } (3/2)^-].$$

- $\Psi \circ X^2$ is not well-defined a priori. But,

$$\Psi \circ X^2 = -I(X^3) \circ X^2 + (u^\dagger \lhd I(X^2)) \circ X^2 + u^\sharp \circ X^2$$

The 3rd term $u^\sharp \circ X^2$ is well-defined. The 1st term $I(X^3) \circ X^2$ is given in the definition of enhancement.

The 2nd term is still ill-defined. By the commutation formula,

$$(u^\dagger \lhd I(X^2)) \circ X^2 = u^\dagger \times (I(X^2) \circ X^2) + \mathrm{Com}(u^\dagger, I(X^2), X^2).$$

The last term is of reg $(1/2)^-$. $I(X^2) \circ X^2$ is given in the enhancement. So, the first term is well-def. and of reg 0^-.

♠ Similarly, we can show $I(\Psi^2 X)$ is of reg. $(3/2)^-$.
♠ Therefore, the integration map Γ can be viewed as a map from $\mathcal{D}_{\vec{\mathcal{X}}}$ to itself.

-For $\mathcal{X} \in C_T\mathcal{C}^{0^+}$, we can take $\vec{\mathcal{X}} = (\mathcal{X}, \mathcal{X}^2, \mathcal{I}(\mathcal{X}^2), \ldots)$ in the literal sense (because $\mathcal{X}$ is a function).
-The solution of the eq. driven by this $\vec{\mathcal{X}}$ coincides with the classical solution of the eq. driven by $\mathcal{X}$.

-Let X^ϵ be a mollified OU process (high frequencies are killed). Take $\vec{\mathcal{X}}^\epsilon$ to be the literal enhancement of X^ϵ.
-You may wish $\vec{\mathcal{X}}^\epsilon$ would converge as $\epsilon \searrow 0$ in $\mathcal{Z}$ (as in RP theory). Unfortunately, it fails!

-Therefore, the enhancement procedure (=the probabilistic part) is quite different from its counterpart in RP theory.

Enhancement/Renormalization of Noise

- $X^\epsilon(t,x)$ is Gaussian and the components of $\vec{\mathcal{X}}^\epsilon$ are its polynomials $\Longrightarrow$ Wiener chaos theory is available.
- By throwing away diverging component (chaos), one can get converging objects.
- The price to pay is that the (S)PDE loses its original form.
- Fortunately, the renormalization constants $c_1^\epsilon, c_2^\epsilon$ are independent of (t,x).

The answer is

$$\Big(X^{\epsilon}, (X^{\epsilon})^2 - c_1^{\epsilon}, I[(X^{\epsilon})^2 - c_1^{\epsilon}], I[(X^{\epsilon})^3 - 3c_1^{\epsilon}X^{\epsilon}], I[(X^{\epsilon})^3 - 3c_1^{\epsilon}X^{\epsilon}] \circ X^{\epsilon},$$
$$I[(X^{\epsilon})^2 - c_1^{\epsilon}] \circ (X^{\epsilon})^2 - c_2^{\epsilon}, \quad I[(X^{\epsilon})^3 - 3c_1^{\epsilon}X^{\epsilon}] \circ (X^{\epsilon})^2 - c_2^{\epsilon}X^{\epsilon}\Big).$$

It converges to a certain limit $(:= \vec{\mathcal{X}}^{\infty})$.

The solution of the generalized SPDE driven by $\vec{\mathcal{X}}^{\infty}$ is what we want.
It is the limit of the solution of the generalized SPDE driven by the deformed noise (due to local well-posedness).

The corresponding classical SPDE is given by

$$\partial_t u^\epsilon = \triangle_x u^\epsilon - (u^\epsilon)^3 + (3c_1^\epsilon + 9c_2^\epsilon)u^\epsilon + \xi \quad \text{with } u(0,\,\cdot\,) = u_0.$$

The renormalization constants $c_1^\epsilon, c_2^\epsilon \nearrow \infty$ as $\epsilon \searrow 0$.

Theorem 1

Assume that u_0 is not bad. Then, there are diverging constants $c_1^\epsilon, c_2^\epsilon$ such that u^ϵ converges (locally in time) to a certain limit u^∞ in $(-1/2)^-$ Besov topology.

Thus, we have obtained a time-local solution of Φ_3^4 equation after renormalization.

Summary: Key ingredients of paracontrolled distribution theory

1. First we have OU process $X = I(\xi)$ whose sample paths belong to $C_T\mathcal{C}^{(-1/2)^-}$.
2. Elements of $C_T\mathcal{C}^{(-1/2)^-}$ are given additional information in a deterministic way ($=$ enhancement). [the deterministic part 1].
3. For each lifted object (i.e., driver), Banach spaces of paracontrolled distributions are defiend so that RHS of Φ^4_3-model makes sense. A solutions is a fixed point in such a Banach spaces. [the deterministic part 2].
4. OU process admits an enhancement a.s. after renormalization $\longrightarrow$ the equation is deformed. [the probabilistic part].

Age of time-global solution

- In GIP theory (and RS theory), only time-local solutions are obtained in general.
- Mourrat-Weber ('16+) proved the global well-posedness by using the special form of Φ^4_3-model. This could be big!

- Consequently, standard problems for SPDEs naturally arise for this model, too. Invariant measure, Dirichlet form, long time problems, (random) dynamical systems, Malliavin calculus, etc.
- Probably, $\exists$ so many others from the physical viewpoint.