Hairer 理論/Gubinelli-Imkeller-Perkowski 理論による Φ^4_3 模型へのアプローチの概説 I

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The 3D dynamic Φ^4-model driven by space-time white noise

Let us study the following real-valued stochastic PDE on $(0,\infty)\times\mathbf{T}^3$, where ξ is the space-time white noise on $\mathbf{R}\times\mathbf{T}^3$ associated with $L^2(\mathbf{R}\times\mathbf{T}^3, dtdx)$ ($\mathbf{T}:=\mathbf{R}/\mathbf{Z}$);

$$\partial_t u = \triangle_x u - u^3 + \xi \qquad \text{with } u(0,\,\cdot\,) = u_0. \tag{1}$$

This is also called *the stochastic quantization equation* and physically very important, but was formerly ill-defined. We consider (generalized) *mild solutions* of this SPDE.

If the nonlinear term u^3 is absent, then the solution is the Ornstein-Uhlenbeck process, whose regularity at a fixed time is $(-1/2)^-$, i.e., $-1/2-\delta$ $(\forall\delta>0)$. One can naturally guess that the regularity of u_t, if it exists, is probably the same at best. It means that u_t is not a function, but merely a distribution and multiplication like u^3 cannot be defined. Therefore, this equation was not solved. More precisely, it was not even well-defined.

First, Hairer solved it a few years ago in his Fields medal awarded paper and soon after that two other methods appeared.

- Hairer's theory of regularity structures [6],
- Gubinelli-Imkeller-Perkowski's paracontrolled calculus, also known as theory of paracontrolled disributions [5],
- Kupiainen's theory based on renormalization group theory [7]. [1]

In this series of two survey talks we discuss recent developments of this stochastic PDE. In the first talk by Y. Inahama, we solve this SPDE via Gubinelli-Imkeller-Perkowski's method. In the second talk by M. Hoshino, we solve this SPDE via Hairer's method. We remark that both theories are descendants of Gubinelli's version of rough path theory.

[1]Nobody in Japan seems to take notice of it. I hope young (or old) folks who are familiar with renormalization groups would take a look at it.

Gubinelli's version of rough path theory
Rough path theory was invented by T. Lyons, but there are now some versions of it.

- Lyons' original rough path theory [8, 10, 9, 3],
- Gubinelli's controlled path theory [4, 2],
- Lyons-Yang's new theory [11], [2] which has no name yet.

The singular SPDE theories we discuss here emerged from the second one, so one must understand or recall it first. In rough path theory, functions (i.e., paths) are defined on a one-dimensional set like $[0, T]$ and the regularity is measured by the Hölder exponents.

Let x be an $\mathbf{R}^d$-valued α-Hölder continuous path. When $\alpha \le 1/2$, an $\mathbf{R}^n$-valued controlled ODE like

$$y_t = y_0 + \int_0^t \sigma(y_s) dx_s$$

does not make sense. Here, σ is a nice function that takes values in the set of $n \times d$-matrices. The reason is heuristically as follows. y is given by a (indefinite) line integral along x, so its regularity is probably the same as that of x, namely α. So is the regularity of $\sigma(y)$. However, $\int \sigma(y) dx$ cannot be defined since the sum of the regulairty of the two path x and $\sigma(y)$ does not satisfy $\alpha + \alpha > 1$, which is the condition for Young integral to hold.

To make sense of such a controlled ODE when $1/3 < \alpha \le 1/2$, a rough path is introduced. It is of the form $(X^1_{s,t}, X^2_{s,t})_{0 \le s \le t \le T}$ with $X^1_{s,t} = x_t - x_s$ with an algebraic constraint called K. T. Chen's identity. The first level path X^1 is essentially the same as x, so some new information, that is X^2, is added to x so that the line integral could be defined. [3]

For each given rough path $X = (X^1, X^2)$, Gubinelli introduced a Banach space of controlled paths. Simply put, a path is controlled by X if its local behavior is similar to (or better than) that of $x = [t \mapsto X^1_{0,t}]$. Therefore, the spaces of controlled paths may be different for different rough paths. [4] The key point of Gubinelli's theory loosely states that if y is controlled by X, then

[2] The authors seem confident, but nobody seems to take notice. I hope young (or old) folks who are familiar with rough paths would take a look at it.

[3] To make something impossible possible, new information must be added.

[4] This is important.

so are $\sigma(y)$ and the line integral $\int \sigma(y)dx$. Not only the line integral can be defined, but it also satisfies reasonable estimates. As a result, a solution of the controlled ODE above is understood as a fixed point of this integration map in the Banach space of controlled paths with respect to X. The solution map (also know as the Lyons-Itô map) is continuous in X and y_0. So far, everything was deterministic and no probability measure was involved.

When we think of applications of rough paths to SDEs like

$$y_t = y_0 + \int_0^t \sigma(y_s) \circ dw_s \qquad \text{(Stratonovich)},$$

probability theory comes in, but *only in the lifting (enhancing)* procedure. Here, (w_t) is the standard d-dimensional Brownian motion. To use rough path theory, we need W^2. A measurable map $w \mapsto (W^1, W^2)$ with the projection onto the first component being the identity is called *a lift or an enhancement* of w. This part cannot be made deterministic. It is not unique, but a canonical choice is $W^{2,ij}_{s,t} = \int_s^t (w^i_u - w^i_s) \circ dw^j_u$. This is called Brownian rough path. If we put it in the Lyons-Itô map, then we get a unique solution of the SDE above (as an image of a continuous map).

To sum up, the rough story of rough path theory is as follows: At the beginning we have the Wiener measure (or Brownian motion) and the usual path space which the Wiener measure lives on (or sample paths of Brownian motion live in). Then;

- Paths in the usual sense are given additional information in a deterministic way (i.e., lift or enhancement). [the deterministic part 1]
- For each lifted object (i.e., rough path), Banach spaces of controlled paths are defiend so that the integral equation under consideration makes sense. A solutions is a fixed point in such a Banach spaces. [the deterministic part 2].
- Brownian motion admits a lift a.s. [the probabilistic part].

In the deterministic parts, the new integration theory of course extends existing ones.

Dynamic Φ^4_3-model via paracontrolled calculus

Paracontrolled calculus was invented in [5]. It was applied to the dynamic Φ^4_3-model by Catellier-Chouk [1]. Unlike the theory of regularity structure, paracontrolled calculus has been gradually improved by many people. Consequently, there is no canonical version.

First we rewrite the dyamanic Φ^4_3-model in the mild form. Let $\triangle$ be the Laplacian on $\mathbf{T}^3$ and $P_t = e^{t\triangle}$ be the corresponding semigroup. For a function (or distribution) $u(t,x)$ defined on $(0,\infty)\times\mathbf{T}^3$, set $I(u)_t = \int_0^t e^{(t-s)\triangle} u_s ds$ (the space-time convolution with the heat kernel). Then, the equation (1) is understood in the mild sense as follows:

$$u_t = P_t u_0 - I(u^3)_t + X_t. \tag{2}$$

Here, $X = I(\xi)$ is the Ornstein-Uhlenbeck process and solves the linearized equation: $\partial_t X_t = \triangle_x X + \xi$. *This Gaussian process X plays the role of Brownian motion in rough path theory.*

For each fixed $t > 0$, the (space) regularity of X_t is $(-1/2)^-$ in the Besov-Hölder sense. One can naturally guess that the regularity of u_t would not be better that that of X_t. Hence, u_t is not a function, but a distribution. This causes a serious trouble since the nonlinear term u_t^3 cannot be defined in the usual sense. (On the other hand, I works for any distribution-valued path fortunately, even if its regularity is very bad).

So, the key question to ask is which kind of information should be added to the "sample path" of X in a deterministic way so that the right hand side of the equation (in particular, u^3) makes sense.

A slightly lengthy, but not very difficult heuristic observation tells us that a possible answer is

$$\bigl(X,\ X^2,\ I(X^2),\ I(X^3),\ I(X^3)\circ X,\ I(X^2)\circ X^2,\ I(X^3)\circ X^2\bigr) \tag{3}$$

with a constraint $(\partial_t - \triangle)I(X^2) = X^2$. This is called a *driver* of Eq. (1). Here, $\circ$ is the resonant term in the *paraproduct theory*, which is similar to the usual multiplication, but its regularity slightly better if it exists. (The resonant term $f\circ g$ exists if and only if the usual multiplication fg exists). As you can easily guess, a driver plays the role of a rough path.

Important remark The symbol X is used in *two senses* in this abstract: X, X^2, $I(X^2)$ etc. in (3) are just coordinates of a generic element of

$$C\bigl([0,T] \to \mathcal{C}^{-1/2-\kappa} \times \mathcal{C}^{-1-\kappa} \times \mathcal{C}^{1-\kappa} \times \mathcal{C}^{1/2-\kappa} \times \mathcal{C}^{-\kappa} \times \mathcal{C}^{-\kappa} \times \mathcal{C}^{-1/2-\kappa}\bigr)$$

$(0 < \kappa \ll 1)$. Here, $\mathcal{C}^\alpha = \mathcal{B}^\alpha_{\infty,\infty}$ stands for the Besov-Hölder space of regularity $\alpha \in \mathbf{R}$. Therefore, X^2 may not mean $X\times X$ in (3) for example. The space of drivers is the closed subset of the above path space with the constraint $(\partial_t - \triangle)I(X^2) = X^2$, which should be understood in the mild senses.

For a given $(X, X^2, \ldots, I(X^3) \circ X^2)$ as in (3), we can actually define Banach spaces of *paracontrolled distributions*. This plays the role of Banach spaces of controlled paths in rough path theory. Besov spaces and paraproducts are used here in an essential way. The right hand side of Eq. (2) makes sense for a paracontrolled distribution u controlled by the driver $(X, X^2, \ldots)$. Loosely speaking, $v = v(t, x)$ is controlled by the driver $(X, X^2, \ldots)$ if there exist $F \in C([0,T] \to \mathcal{C}^{1/2-\kappa})$ and $G \in C([0,T] \to \mathcal{C}^{3/2-\kappa})$ such that

$$v_t = I(X^3)_t + F_t \lhd I(X^2)_t + G_t. \tag{4}$$

Here, $\lhd$ stands for the *paraproduct* of F_t and $I(X^2)_t$.

A rough and heauristic meaning of (4) is as follows: v_t is of regualrity $(1/2)^-$. The first (i.e., coarsest) approximation of v_t is given by $I(X^3)_t$ whose regualrity is $(1/2)^-$, too. The difference $v_t - I(X^3)_t$ is of better regualrity 1^-. This difference should behave like $I(X^2)_t$ in small scales. (Note that small scale behavior of $F_t \lhd I(X^2)_t$ is similar to that of $I(X^2)_t$.) If $F_t \lhd I(X^2)_t$ is subtracted from $v_t - I(X^3)_t$, then regualrity is $(3/2)^-$. In other words, $v_t - I(X^3)_t$ is allowed to have a bad term (a term of regularity less than $(3/2)^-$) only if it behaves like $I(X^2)_t$. If a term of $v_t - I(X^3)_t$ does not look like $I(X^2)_t$, then it must have better regularity $(3/2)^-$.

A solution of Eq. (2) is defined to be a fixed point in an appropriate space of paracontrolled distribution. Under mild assumptions, well-posedness of time-local solution can be proven. This is the determistic part of this theory.

Of course, this extends the existing theory. Suppose that X is very nice, for example, X is a deterministic element in $C([0,T] \to \mathcal{C}^\alpha)$ for some $\alpha > 0$. In this case, we can choose $\big(X, X^2, \ldots, I(X^3) \circ X^2\big)$ in *the literal sense* (namely, $X^2 := X \times X$, etc.). Then, a unique solution of the new extended equation coincides with the one in the usual sense.

Next we discuss the probabilistic part of the theory, that is, enhancement of the Ornstein-Uhlenbeck process. This part becomes much more complicated than the corresponding part in rough path theory since we need to do some kind of *renormalization*.

Let $X = I(\xi)$ be an Ornstein-Uhlenbeck process again. Since we cannot enhance X directly, we consider a mollified noise X^ε at first. (High frequencies are killed. As $\varepsilon \searrow 0$, $X^\varepsilon \to X$ in an appropriate sense). Since sample paths of X^ε are very nice, we can do the "literal enhancement" of X^ε as above. Unfortunately, however, $\big(X^\varepsilon, (X^\varepsilon)^2, \ldots, I((X^\varepsilon)^3 \circ (X^\varepsilon)^2)\big)$ does not converge! Hence, we cannot get a decent object in this way.

Observe that each component of the above enhanced noise belongs to an inhomogeneous Wiener chaos (at least for fixed ε, t and x). Fortunately, the top order terms of the Wiener chaos expansion are all convergent, though some lower order terms diverge. So, we can throw away these diverging terms in a systematic way by using Wiener chaos theory to get a meaningful limiting object on the space of drivers.

In this way we get a kind of SPDE driven by this limiting object. This procedure is called *renomalization*. However, we have to pay a price for the renomalization. The original form of SPDE is lost. We prefer convergence of the enhanced noise to keeping the original form of the SPDE.

More precisely, there exists diverging real constants c_1^ε and c_2^ε (independent of t and x) such that

$$\Big(X^\varepsilon,\ (X^\varepsilon)^2 - c_1^\varepsilon,\ I((X^\varepsilon)^2 - c_1^\varepsilon),\ I((X^\varepsilon)^3 - c_1^\varepsilon X^\varepsilon),\ I((X^\varepsilon)^3 - c_1^\varepsilon X^\varepsilon) \circ X^\varepsilon,$$
$$I((X^\varepsilon)^2 - c_1^\varepsilon) \circ ((X^\varepsilon)^2 - c_1^\varepsilon) - c_2^\varepsilon,\ I((X^\varepsilon)^3 - c_1^\varepsilon X^\varepsilon) \circ ((X^\varepsilon)^2 - c_1^\varepsilon) - c_2^\varepsilon X^\varepsilon\Big)$$

converges in the space of drivers. The limit is denoted by

$$\Big(X^\infty, (X^\infty)^2, \ldots, I((X^\infty)^3) \circ (X^\infty)^2\Big).$$

Therefore, we get a generalized (S)PDE driven by the above random drivers.

However, since the noise is deformed, it is not clear what this new (S)PDE looks like. In this case, fortunately, it is not so hard see that a unique solution of the new generalized (S)PDE driven by the deformed noise $(X^\varepsilon, (X^\varepsilon)^2 - c_1^\varepsilon, \ldots)$ solves the following (S)PDE in the usual sense:

$$\partial_t u^\varepsilon = \triangle_x u^\varepsilon - (u^\varepsilon)^3 + (3c_1^\varepsilon + 9c_2^\varepsilon)u^\varepsilon + \xi^\varepsilon, \qquad \text{with } u^\varepsilon(0, \cdot\,) = u_0.$$

Observe that the first order term $(3c_1^\varepsilon + 9c_2^\varepsilon)u^\varepsilon$ appeared due to the renormalization.

In summary, we have the following result. *If the initial value u_0 is not so bad, then there exists a random time $T_* > 0$ such that u^ε converges to a certain limit u^∞ on the time interval $[0, T_*)$ in an appropriate Banach space of paracontrolled distributions.*

Some remarks are in order. (i) The limit u^∞ may not solve any (S)PDE in the usual sense. But, it certainly is a solution of the new generalized (S)PDE driven by a random driver $(X^\infty, (X^\infty)^2, \ldots)$. So, it is not very strange to call it a solution of an SPDE.

(ii) In their recent work, Mourrat and Weber [12] proved this equation in fact has a time-global solution for any driver in a deterministic sense. (In my view, this could be a breakthrough.) Moreover, their method is new. Without defining the spaces of paracontrolled distributions, they directly decompose Eq. (1) into a system of two PDEs, the first one of which is a linear equation involving the paraproduct with respect to $I(X^2)$.

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