Rough differential equations containing path-dependent bounded variation terms

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Let E be a finite dimensional normed linear space. For a continuous path (w_t) $(0 \le t \le T)$ on E, we define for $[s,t] \subset [0,T]$,

$$\|w\|_{\infty-var,[s,t]} = \max_{s \le u \le v \le t} |w_{u,v}|, \tag{1}$$

$$\|w\|_{p-var,[s,t]} = \left\{ \sup_{\mathcal{P}} \sum_{k=1}^{N} |w_{t_{k-1},t_k}|^p \right\}^{1/p}, \tag{2}$$

where $\mathcal{P} = \{s = t_0 < \cdots < t_N = t\}$ is a partition of the interval $[s,t]$ and $w_{u,v} = w_v - w_u$. When $[s,t] = [0,T]$, we may omit denoting $[0,T]$.

Let $\omega(s,t)$ $(0 \le s \le t \le T)$ be a control function. That is, $(s,t) \mapsto \omega(s,t) \in \mathbb{R}^+$ is a continuous function and $\omega(s,u) + \omega(u,t) \le \omega(s,t)$ $(0 \le s \le u \le t \le T)$ holds. We introduce mixed norms by using ω and p-variation norm. For $0 < \theta \le 1, q \ge 1$, $0 \le s \le t \le T$ and a continuous path w, we define

$$\|w\|_{\theta,[s,t]} = \inf \left\{ C > 0 \mid |w_{u,v}| \le C\omega(u,v)^{\theta} \quad s \le u \le v \le t \right\}, \tag{3}$$

$$\|w\|_{q-var,\theta,[s,t]} = \inf \left\{ C > 0 \,\middle|\, \|w\|_{q-var,[u,v]} \le C\omega(u,v)^{\theta} \quad s \le u \le v \le t \right\}. \tag{4}$$

When $\omega(s,t) = |t-s|$, $\|w\|_{\theta,[s,t]} < \infty$ is equivalent to that w_u $(s \le u \le t)$ is a Hölder continuous path with the exponent θ in usual sense. Hence we may say w is an ω-Hölder continuous path with the exponent θ ((ω,θ)-Hölder continuous path in short). For two parameter function $F_{s,t}$ $(0 \le s \le t \le T)$, we define $\|F\|_{\theta,[s,t]}$ and $\|F\|_{q-var,\theta,[s,t]}$ similarly.

Let $\mathcal{V}_{q,\theta,T}(E)$ denote the set of E-valued continuous paths of finite q-variation defined on $[0,T]$ satisfying $\|w\|_{q-var,\theta} := \|w\|_{q-var,\theta,[0,T]} < \infty$. Note that $\mathcal{V}_{q,\theta,T}(E)$ is a Banach space with the norm $|w_0| + \|w\|_{q-var,\theta}$. Obviously, any path $w \in \mathcal{V}_{q,\theta,T}$ satisfy $|w_{s,t}| \le \|w\|_{q,\theta}\omega(s,t)^{\theta}$.

We denote by $\mathcal{V}_{\theta}$ the set of ω-Hölder continuous paths w satisfying $\|w\|_{\theta} = \|w\|_{\theta,[0,T]} < \infty$. $\mathcal{V}_{\theta}$ is a Banach space with the norm $|w_0| + \|w\|_{\theta}$.

Let $1/3 < \beta \le 1/2$. Let $\mathbf{X}_{s,t} = (X_{s,t}, \mathbb{X}_{s,t})$ $(0 \le s \le t \le T)$ be a $1/\beta$-rough path on $\mathbb{R}^n$ with the control function ω. That is, $\mathbf{X}$ satisfies Chen's relation and the path regularity conditions,

$$|X_{s,t}| \le \|X\|_{\beta}\omega(s,t)^{\beta}, \quad |\mathbb{X}_{s,t}| \le \|\mathbb{X}\|_{2\beta}\omega(s,t)^{2\beta}, \qquad 0 \le s \le t \le T. \tag{5}$$

We denote by $\mathscr{C}^{\beta}(\mathbb{R}^n)$ the set of $1/\beta$-rough paths. When $\omega(s,t) = |t-s|$, $\mathbf{X}_{s,t}$ is a β-Hölder rough path. If $\mathbf{X}_{s,t}$ is a rough path with finite $1/\beta$-variation, setting $\omega(s,t) = \|X\|_{1/\beta-var,[s,t]}^{1/\beta} + \|\mathbb{X}\|_{2/\beta-var,[s,t]}^{2/\beta}$, we have $\|X\|_{\beta} = \|\mathbb{X}\|_{2\beta} = 1$.

Let us choose p and γ such that $2 \le 1/\beta < p < \gamma \le 3$. We use the following quantity,

$$\widetilde{\|\!|\mathbf{X}|\!\|_\beta} = \sum_{i=1}^{3} \|\!|\mathbf{X}|\!\|_\beta^i, \quad \|\!|\mathbf{X}|\!\|_\beta = \|\mathbf{X}\|_\beta + \sqrt{\|\mathbb{X}\|_{\mathbf{2}\beta}}. \tag{6}$$

We introduce a set of controlled paths $\mathscr{D}_X^{2\theta}(\mathbb{R}^d)$ of $\mathbf{X}_{s,t}$, where $1/3 < \theta \le \beta$. A pair of ω-Hölder continuous paths $(Z, Z') \in \mathcal{V}_\theta([0,T], \mathbb{R}^d) \times \mathcal{V}_\theta([0,T], \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$ with the exponent θ is called a controlled path of X, if the remainder term $R^Z_{s,t} = Z_t - Z_s - Z'_s X_{s,t}$ satisfies $\|R^Z\|_{2\theta} < \infty$. The set of controlled paths $\mathscr{D}_X^{2\theta}(\mathbb{R}^d)$ is a Banach space with the norm

$$\|(Z, Z')\|_{2\theta} = |Z_0| + |Z'_0| + \|Z'\|_\theta + \|R^Z\|_{2\theta} \quad (Z, Z') \in \mathscr{D}_X^{2\theta}(\mathbb{R}^d) \tag{7}$$

$Z'_t \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$ is called a Gubinelli derivative of Z with respect to X.

The rough differential equation which we will study contains path dependent bounded variation term $L(w)_t$. We consider the following condition on L.

Assumption 1. *Let $\xi, \eta \in \mathbb{R}^d$. Let L be a mapping from $\mathcal{V}_\beta([0,T] \to \mathbb{R}^d \mid w_0 = \xi)$ to $C([0,T] \to \mathbb{R}^d \mid w_0 = \eta)$ and satisfy the following conditions.*

(1) *(adaptedness) $(L(w)_s)_{0 \le s \le t}$ depends only on $(w_s)_{0 \le s \le t}$ for all $0 \le t \le T$.*

(2) *$L : (\mathcal{V}_\beta, \|\cdot\|_{\beta'}) \to (C([0,T]), \|\cdot\|_{\infty-var})$ is continuous for some $\beta' < \beta$.*

(3) *There exists a non-decreasing positive continuous function F on $[0,\infty)$ such that*

$$\|L(w)\|_{1-var,[s,t]} \le F(\|w\|_{(1/\beta)-var,[s,t]}) \|w\|_{\infty-var,[s,t]}. \tag{8}$$

We now state our main theorem. Note that we need to give the precise meaning of the integral below. We will do so in the talk.

Theorem 2. *Let $\sigma \in \mathrm{Lip}^{\gamma-1}(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$ and $\xi, \eta \in \mathbb{R}^d$. Assume that the mapping $L : \mathcal{V}_\beta([0,T] \to \mathbb{R}^d \mid w_0 = \xi) \to C([0,T] \to \mathbb{R}^d \mid w_0 = \eta)$ satisfies the condition in Assumption 1. Then there exists a controlled path $(Z, Z') \in \mathscr{D}_X^{2\beta}(\mathbb{R}^d)$ such that*

$$Z_t = \xi + \int_0^t \sigma(Z_s, L(Z)_s) d\mathbf{X}_s, \tag{9}$$

$$Z'_t = \sigma(Z_t, L(Z)_t) \tag{10}$$

Further there exist positive constants κ, C_1, C_2, C_3 which depend only on σ, β, p, γ such that

$$\begin{aligned} &\|Z\|_\beta + \|R^Z\|_{2\beta} + \|L(Z)\|_{1-var,\beta} \\ &\le C_1 \Big\{ 1 + \Big(1 + F(C_2 \widetilde{\|\!|\mathbf{X}|\!\|_\beta})\Big)^\kappa \Big(1 + \widetilde{\|\!|\mathbf{X}|\!\|_\beta}\Big)^\kappa \omega(0,T) \Big\} \Big(1 + F(C_3 \widetilde{\|\!|\mathbf{X}|\!\|_\beta})\Big) \widetilde{\|\!|\mathbf{X}|\!\|_\beta}. \end{aligned} \tag{11}$$

Remark 3 (Reflected rough differential equations)**.** This theorem implies the existence of solutions of reflected rough differential equations under the famous conditions (A) and (B) of the boundary. This is an extension of the speaker's result in SPA 125 (2015). Note that a stronger condition (H1) was imposed on the boundary in the previous paper. We do not need such a condition in this new approach.

J. Ren and J. Wu (Ann. Probab. 44 (2016)) proved a support theorem for reflected diffusions under the conditions (A), (B) and (C) by using the Wong-Zakai type theorem (A-Sasaki, SPA 123, 2013). We can give another proof of the support theorem by using the above theorem and the Wong-Zakai type theorem under the conditions (A) and (B).