SLK martingales and representations of the Witt algebra

Kazuhiro Yoshikawa (Ritsumeikan University)
Takafumi Amaba * (Ritsumeikan University)

1. Introduction

The Loewner differential equation whose driving function is a Brownian motion is called the Schramm Loewner evolution (SLE). The random coefficients of the expansion of the SLE have a hierarchy of stochastic differential equations, which induces a class of polynomials characterized by martingales. It is known that those polynomials connect the SLE to representations of the Virasoro algebra ([3]). In this talk, we introduce random coefficients based on the Loewner-Kufarev equation and martingales related with the Kirillov-Neretin polynomials. Our aim is to find some relations between stochastic differential equations and representations of the Virasoro algebra

2. A hierarchical solution of stochastic differential equations

Put $D = \{z \in \mathbb{C} \mid |z| < 1\}$. We consider stochastic processes $C(t), c_1(t), c_2(t), \ldots$ generated by a holomorphic function $g_t(z)$ on D :

$$g_t(z) = C(t)(z + c_1(t)z^2 + c_2(t)z^3 + \cdots)$$

which is a solution of the following stochastic differential equation

$$\begin{cases} dg_t(z) = zg_t'(z)\Big\{dX_t^0 + \sum_{k=1}^{\infty} z^k dX_t^k\Big\}, \\ g_0(z) = z \in D. \end{cases} \tag{2.1}$$

Here $X_t^0 = \alpha_0^{-1}t$, $\alpha_0 > 0$, $X_t^k = \alpha_k^{-1}Z_t^k$, $\alpha_k > 0$ for $k \geq 1$ and $Z_t^1, Z_t^2, \ldots$ are infinitely many independent complex Brownian motions. We note that the solution of (2.1) means a hierarchy of stochastic differential equations for the coefficients of the expansion of $g_t(z)$:

$$\begin{cases} dC(t) &= C(t)dX_t^0, \\ dc_1(t) &= dX_t^1 + c_1(t)dX_t^0, \\ dc_2(t) &= dX_t^2 + 2c_1(t)dX_t^1 + 2c_2(t)dX_t^0, \\ dc_3(t) &= dX_t^3 + 2c_1(t)dX_t^2 + 3c_2(t)dX_t^1 + 3c_3(t)dX_t^0, \\ &\vdots \end{cases}$$

In [2], we regarded this hierarchy as a solution of a stochastic Lowener Kufarev equation, which is an approach for constructions of measures on loops.

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3. The Kirillov-Neretin polynomials

We denote by $\mathcal{M}$ the set of all holomorphic functions $f : \bar{D} \to \mathbb{C}$ written as

$$f(z) = z(1 + \sum_{n=1}^{\infty} c_n z^n) \quad \text{for all } z \in D.$$

The Kirillov-Neretin polynomials can be defined as follows:

$$\sum_{n=0}^{\infty} P_n(c_1, \dots, c_n) z^n = h\Big(\frac{zf'(z)}{f(z)}\Big)^2 + \frac{cz^2}{12} S(f)(z), \quad \text{for all } f \in \mathcal{M},$$

where $S(f)$ is the Schwarzian derivative of f :

$$S(f)(z) := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2.$$

In particular, if $h = 0$, then we have

$$P_0 = P_1(c_1) = 0, \quad P_2(c_1, c_2) = \gamma_2(c_2 - c_1^2), \quad P_3(c_1, c_2, c_3) = \gamma_3(c_3 - 2c_1c_2 + c_1^3),$$
$$P_4(c_1, c_2, c_3, c_4) = \gamma_4\big(c_4 - 2c_1c_3 - \frac{6}{5}c_2^2 + \frac{17}{5}c_1^2c_2 - \frac{6}{5}c_1^4\big), \dots,$$

where $\gamma_k := \frac{c}{12}(k^3 - k)$. We can find the formula for Virasoro algebra to act the Kirillov-Neretin polynomials in [1], [4].

4. Martingales based on the Kirillov-Neretin polynomials

Now we put

$$P_n(t) := P_n(c_1(t), \dots, c_n(t)), \quad n \in \mathbb{Z}_{\geq 0}.$$

The first few terms of the stochastic processes $P_n(t)$ are as follows:

$$\begin{aligned}
dP_2(t) &= \gamma_2 dX_t^2 + 2P_2(t)dX_t^0, \\
dP_3(t) &= \gamma_3 dX_t^3 + 4P_2(t)dX_t^1 + 3P_3(t)dX_t^0, \\
dP_4(t) &= \gamma_4 dX_t^4 + 6P_2(t)dX_t^2 + 5P_3(t)dX_t^1 + 4P_4(t)dX_t^0.
\end{aligned}$$

Then, we gain the following result.

Theorem 4.1. *For all $n = 0, 1, 2, \dots$,*

$$e^{-nt/\alpha_0} P_n(t) \textit{ is a (local) martingale.}$$

Moreover, these martingales are generated by successive actions of the Witt algebra.

References

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