On the derivation of noncausal function from its Haar-SFCs *

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(i) SFC. Let $f(t,\omega)$ be a random function on $[0,1]\times\Omega$ and $\{\varphi_n(t)\}$ be a CONS in $L^2([0,1];\mathbb{C})$. The system $\{\hat{f}_n(\omega) = \int_0^1 f(t,\omega)\overline{\varphi_n(t)}dW_t\}$ is called the *stochastic Fourier coefficients* (*SFCs* in abbr.) of $f(t,\omega)$, $\{W(t), t\in[0,1]\}$ being a Brownian motion on $(\Omega,\mathcal{F},P)$ which starts at the origin. It is of course that the stochastic integral $\int dW$ in the definition of SFCs should adequately be chosen according to the conditions on $f(t,\omega)$. We are concerned with the problem whether $f(t,\omega)$ is identified from the SFCs of $f(t,\omega)$ or not.

In this talk we consider the case that $f(t,\omega)$ is noncausal, and we aimed to identify $f(t,\omega)$ without the aid of a Brownian motion. Moreover, we intend to derive $f(t,\omega_0)$ from SFCs $\{\hat{f}_n(\omega_0)\}$ for almost all ω_0.

(ii) SFT. Let $\{\varepsilon_n\}$ be an ℓ_2 sequence such that $\varepsilon_n \neq 0$ for all n. Then

$$\mathcal{T}_{(\varepsilon,\varphi)}(f)(t,\omega) = \sum_n \varepsilon_n \hat{f}_n(\omega)\varphi_n(t)$$

is called $(\varepsilon_n,\varphi_n)$-*stochastic Fourier transform* (*SFT* in abbr.) of $f(t,\omega)$. In [1] we discussed this problem under the condition that SFCs are defined by employing the Ogawa integral as a stochastic integral and the system of trigonometric functions $e_n(t) = \mathrm{e}^{2\pi int}$, $n\in\mathbb{Z}$, as a CONS. We assumed the next three conditions on $f(t,\omega)$;

[H1] For almost all ω, $f(t,\omega)$ is a differentiable function with respect to t satisfying $f'(t,\omega)\in L^2([0,1],dt)$, where $f'(t,\omega) = \partial f(t,\omega)/\partial t$.

[H2] $\int_0^1 f(t,\omega)dt \in L^2(\Omega,dP)$ and $f'(t,\omega)\in L^2([0,1]\times\Omega, dtdP)$.

[H3] For almost all ω, $f(t,\omega)$ is a nonnegative function.

[H.1] assures us of the existence of SFCs, and the (τ_n,e_n)-SFT $\mathcal{T}_{(\tau,e)}(f)(t,\omega)$ of $f(t,\omega)$ exists in $C^1(0,1)$ under the condition [H.2], where $\tau_n = (-4\pi^2n^2)^{-1}$ if $n\neq 0$ and $\tau_0 = 1$. From [H.3] and the law of iterated logarithm of the Brownian motion we have

$$P\left(\limsup_{h\downarrow 0}\frac{\mathcal{T}_{(\tau,e)}(f)'(t+h,\omega)-\mathcal{T}_{(\tau,e)}(f)'(t,\omega)}{\sqrt{2h\log\log\frac{1}{h}}} = f(t,\omega),\quad \forall t\in\mathbb{T}\right) = 1,$$

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where $\mathbb{T}$ is an arbitrary dense subset of $(0,1)$.

(iii) Haar-SFC. In this talk we employ the Ogawa integral and the system of Haar functions to define SFCs of $f(t,\omega)$. We assume the next two conditions on $f(t,\omega)$;

[H1'] For almost all ω, $f(t,\omega)$ is a continuous function on $[0,1]$ satisfying there exists a function $g(s,\omega) \in L^2([0,1],ds)$ such that $f(t,\omega) - f(0,\omega) = \int_0^t g(s,\omega)ds$.

[H3] For almost all ω, $f(t,\omega)$ is a nonnegative function.

Let $\{H_k^{(n)}; (n,k) \in \Lambda\}$ be the system of Haar functions on $[0,1]$, i.e., $H_0^{(0)}(t) = 1$ and

$$H_k^{(n)}(t) = \begin{cases} 2^{(n-1)/2} & (t_{n.2k} \leq t < t_{n.2k+1}) \\ -2^{(n-1)/2} & (t_{n.2k+1} \leq t < t_{n.2k+2}) \\ 0 & (\text{otherwise}) \end{cases}$$
$$(n = 1,2,\ldots,\ k = 0,1,\ldots,2^{n-1}-1),$$

where $t_{n,k} = k/2^n$. We denote the Haar SFC corresponding to $H_k^{(n)}(t)$ by $\hat{f}_k^{(n)}(\omega)$:

$$\hat{f}_k^{(n)}(\omega) = \int_0^1 f(t,\omega) H_k^{(n)}(t) d_* W_t,$$

$\int d_* W_t$ denoting the Ogawa integral. Set

$$S_N(t.\omega) = \hat{f}_0^{(0)}(\omega) H_0^{(0)}(t) + \sum_{n=1}^{N} \sum_{k=0}^{2^{n-1}-1} \hat{f}_k^{(n)}(\omega) H_k^{(n)}(t).$$

Then we have the following lemma;

Lemma 1. *For $t \in [t_{N,\ell}, t_{N,\ell+1})$, $\ell = 0,1,\ldots,2^N-1$, it holds that*

$$S_N(t.\omega) = 2^N \left[f(t_{N,\ell+1},\omega) W(t_{N,\ell+1}) - f(t_{N,\ell},\omega) W(t_{N,\ell}) - \int_{t_{N,\ell}}^{t_{N,\ell+1}} g(t,\omega) W(t) dt \right].$$

From [H.3] and the law of iterated logarithm of the Brownian motion we have our main theorem;

Theorem 1. *Suppose that $f(t,\omega)$ satisfies conditions [H.1'] and [H.3]. Let $\mathbb{T}$ be a countable dense subset of $[0,1)$. Then we have*

$$P\left(\limsup_{N\to\infty} \frac{S_N(t,\omega)}{\sqrt{2^{N+1}\log N}} = f(t,\omega), \quad \forall t \in \mathbb{T} \right) = 1.$$

References

[1] Ogawa,S and Uemura,H.: "On the identification of noncausal functions from the SFCs", RIMS Kôkyûroku 1952 (2015), 128–134