Parametrix method for skew diffusions

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Outline

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Skew diffusion

• A skew diffusion is the unique solution of the following one-dimensional stochastic differential equation with symmetric local time:

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + (2\alpha - 1)L_t^0(X), \qquad (1)$$

where $t \in [0, T]$ and $|2\alpha - 1| \in (0, 1)$.

• $W = (W_t)_{0 \le t \le T}$ is a one-dimensional standard Brownian motion.

• $L^0(X) = (L_t^0(X))_{0 \le t \le T}$ is a symmetric local time of X at the origin, that is

$$L_t^0(X) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[-\varepsilon,\varepsilon]}(X_s)d\langle X\rangle_s.$$

• If $\alpha = 1/2$, then a solution to the equation (1) is a diffusion process.

• If $\alpha = 1$ or $\alpha = 0$, then a solution to the equation (1) is reflected stochastic differential equation.

• We want to prove that

(i) Existence of the density of a skew diffusion.

(ii) The density of a skew diffusion satisfies a Gaussian upper bound.

• We want to calculate an expectation $\mathbb{E}[g(X_T)]$ for some function g.

• Application: Physics, PDE and SDEs with dis-continuous coefficients.

Relation to SDEs with dis-continuous coefficients

Define

$$s_\alpha(x) := (1-\alpha)x\mathbf{1}(x \geq 0) + \alpha x \mathbf{1}(x < 0).$$

By using the symmetric Itô-Tanaka formula, we have

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + (2\alpha - 1)L_t^0(X)$$

$$s_\alpha \downarrow \uparrow s_\alpha^{-1}$$

$$Z_t := s_\alpha(X_t) = s_\alpha(x) + \int_0^t \mu(Z_s)ds + \int_0^t \rho(Z_s)dW_s,$$

where

$$\mu(z) := (1-\alpha)b\left(\frac{z}{1-\alpha}\right)\mathbf{1}(z>0) + \alpha b\left(\frac{z}{\alpha}\right)\mathbf{1}(z<0) + \frac{b(0)}{2}\mathbf{1}(z=0),$$

$$\rho(z) := (1-\alpha)\sigma\left(\frac{z}{1-\alpha}\right)\mathbf{1}(z>0) + \alpha \sigma\left(\frac{z}{\alpha}\right)\mathbf{1}(z<0) + \frac{\sigma(0)}{2}\mathbf{1}(z=0).$$

Weak solution: Krylov [6], Strong solution: Le Gall [7], Nakao [8].

Main result

Let $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Our first main result on this talk is the following.

Theorem 1

Assume that

(i) σ *is a positive, bounded and uniformly elliptic function. In particular, there exist positive constants* $\overline{a}$ *and* $\underline{a}$*, such that for any* $x \in \mathbb{R}$*,*

$$\underline{a} \leq a(x) := \sigma^2(x) \leq \overline{a}.$$

(ii) b *is bounded measurable and* a *is* η*-Hölder continuous with* $\eta \in (0, 1]$*, i.e., there exist positive constant* K *such that*

$$\sup_{x\in\mathbb{R}} |b(x)| + \sup_{x,y\in\mathbb{R},x\neq y} \frac{|a(x) - a(y)|}{|x - y|^\eta} \leq K.$$

Then for any $(t, x) \in (0, T] \times \mathbb{R}_0$*, there exists the density function of* $X_t(x)$*,* $p_t(x, \cdot)$*, which satisfies the Gaussian upper bound, i.e., there exist positive constants* C *and* c *such that, for any* $(t, x, y) \in (0, T] \times \mathbb{R}_0 \times \mathbb{R}$*,*

$$p_t(x, y) \leq \frac{Ce^{-\frac{(y-x)^2}{2ct}}}{\sqrt{2\pi ct}}.$$

Main result

Moreover, the density $p_t(x, \cdot)$ is a differentiable with respect to an initial value $x \in \mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ and satisfies the following conditions:

$$\partial_x p_t(x, y) \leq \frac{C}{t^{1/2}} \frac{e^{-\frac{(y-x)^2}{2ct}}}{\sqrt{2\pi c t}},$$

and

$$\alpha \partial_x p_t(0+, y) = (1-\alpha) \partial_x p_t(0-, y), \tag{2}$$

and if $\alpha \neq 1/2$, $p_t(x, \cdot)$ is discontinuous at zero.

Remark 1

Note that the property (2) implies that if $\alpha \neq 1/2$, then the density function of skew diffusion cannot differentiable at zero with respect to x.

Parametrix method for diffusion

In this section, we introduce a parametrix method for one-dimensional diffusion process:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x.$$

A parametrix method is a “Taylor-like expansion" for the density of diffusion process and is used to construct a fundamental solution for parabolic type PDEs (Levi or Friedman [2])

For solution of SDE X, the infinitesimal generator L is given by

$$Lf(x) = \frac{a(x)}{2}f''(x) + b(x)f'(x),\ f \in C^2$$

For simplicity, we assume that $b, \sigma \in C_b^\infty$ and σ is uniformly elliptic. Then there exists the density function $p_t(x, \cdot)$ of X_t satisfying

$$\partial_t p_{t-s}(x,y) = L^* p_{t-s}(x,y), \ \lim_{t \downarrow s} \int_{\mathbb{R}} f(x) p_{t-s}(x,y) dx = f(y)$$

$$\partial_s p_{t-s}(x,y) = -L p_{t-s}(x,y), \ \lim_{t \uparrow s} \int_{\mathbb{R}} f(y) p_{t-s}(x,y) dy = f(x).$$

Consider a “frozen process" (approximation process)

$$X_t^y = x + b(y)t + \sigma(y) W_t, \text{ (or } X_t^y = x + \sigma(t) W_t).$$

Let $p_t^y(x,.)$ be a density function of X_t^y. Then $\overline{p}(x,y) := p_t^y(x,y)$, satisfies

$$\partial_s \overline{p}_{t-s}(x,y) = -L^y \overline{p}_{t-s}(x,y), \ \lim_{t \downarrow s} \int_{\mathbb{R}} f(x) \overline{p}_{t-s}(x,y) dx = f(y),$$

where

$$L^y f(x) = \frac{a(y)}{2} f''(x) + b(y) f'(x), \text{ (or } \frac{a(y)}{2} f''(x)).$$

Hence we have

$$\begin{aligned}
p_t(x,y) - \overline{p}_t(x,y) &= \int_0^t ds\partial_s \int_{\mathbb{R}} dz p_s(x,z)\overline{p}_{t-s}(z,y) \\
&= \int_0^t ds \int_{\mathbb{R}} dz \left(\partial_s p_s(x,z)\overline{p}_{t-s}(z,y) + p_s(x,z)\partial_s \overline{p}_{t-s}(z,y)\right) \\
&= \int_0^t ds \int_{\mathbb{R}} dz \left(L^* p_s(x,z)\overline{p}_{t-s}(z,y) - p_s(x,z)L^y \overline{p}_{t-s}(z,y)\right) \\
&= \int_0^t ds \int_{\mathbb{R}} dz p_s(x,z) \underbrace{(L - L^y)\bar{p}_{t-s}(z,y)}_{=:\Phi_{t-s}(z,y)} \\
&= \int_0^t ds \int_{\mathbb{R}} dz p_s(x,z)\Phi_{t-s}(z,y) \\
&=: p \circledast \Phi(t,x,y).
\end{aligned}$$

This implies that

$$p_t(x,y) = \overline{p}_t(x,y) + p \circledast \Phi(t,x,y).$$

$\overline{p}_t(x,y)$ is called the “parametrix" and this procedure is called the “parametrix method".

By iterating the above procedure, we have the following “formal expansion"

$$
\begin{aligned}
p_t(x,y) &= \overline{p}_t(x,y) + p \circledast \Phi(t,x,y) \\
&= \overline{p}_t(x,y) + \overline{p} \circledast \Phi(t,x,y) + p \circledast \Phi^{\circledast 2}(t,x,y) \\
\text{“} &= \text{”} \sum_{n=0}^{\infty} \overline{p} \circledast \Phi^{\circledast n}(t,x,y).
\end{aligned} \tag{3}
$$

Under the following Assumption, the above expansion holds.

Assumption 1

We assume that the drift coefficient b and diffusion coefficient σ satisfy the following conditions:

- (B) b *and* σ *are bounded and measurable.*
- (UE) σ *is a positive, bounded and uniformly elliptic function. In particular, there exist positive constants* $\overline{a}$ *and* $\underline{a}$*, such that for any* $x \in \mathbb{R}$, $\underline{a} \leq a(x) := \sigma^2(x) \leq \overline{a}$.
- (η-H) a *is* η*-Hölder continuous with* $\eta \in (0,1]$*. That is, there exists* $K > 0$ *such that* $\sup_{x \neq y} \frac{|a(x)-a(y)|}{|x-y|^{\eta}} \leq K$.

Bally and Kohatsu-Higa [1] introduce parametrix expansion for semigroup

Moreover, under Assumption 1, the following Gaussian upper bound holds:

$$p_t(x,y) \le \sum_{n=0}^{\infty} |\overline{p} \circledast \Phi^{\circledast n}(t,x,y)| \le \sum_{n=0}^{\infty} \frac{C^n}{\Gamma(1+n\eta/2)} \frac{e^{-\frac{(y-x)^2}{2ct}}}{\sqrt{2\pi ct}} = \frac{Ce^{-\frac{(y-x)^2}{2ct}}}{\sqrt{2\pi ct}}.$$

Note that it is well-known that if b is also Hölder continuous, we can prove that $p_t(x,y)$ satisfies the following PDE

$$\partial_s p_{t-s}(x,y) = -Lp_{t-s}(x,y), \ \lim_{t \uparrow s} \int_{\mathbb{R}} f(y)p_{t-s}(x,y)dy = f(x).$$

Why Hölder continuous ?

A Hölder continuity of $a = \sigma^2$ gives us integrability with respect to times variables, that is

$$\frac{|a(x) - a(y)|}{t}\frac{e^{-\frac{|x-y|^2}{2t}}}{\sqrt{2\pi t}} \leq \frac{C}{t^{1-\eta/2}}\underbrace{\frac{|x-y|^{\eta}}{t^{\eta/2}}e^{-\frac{1}{2}\frac{|x-y|^2}{2t}}}_{\leq C}\frac{e^{-\frac{1}{2}\frac{|x-y|^2}{2t}}}{\sqrt{2\pi t}} \leq \frac{C}{t^{1-\eta/2}}\frac{e^{-\frac{1}{2}\frac{|x-y|^2}{2t}}}{\sqrt{2\pi t}},$$

Hence, we have

$$\int_0^T dt \int_{\mathbb{R}} dy \frac{|a(x) - a(y)|}{t}\frac{e^{-\frac{|x-y|^2}{2t}}}{\sqrt{2\pi t}} < \infty.$$

This is the reason, why the parametrix expansion convergences and Gaussian upper bound holds.

Parametrix method for skew diffusion

Recall that for the parametrix method for diffusion process, a “frozen process" X^y is defined by

$$X_t^y = x + b(y)t + \sigma(y)W_t, \text{ (or } X_t^y = x + \sigma(y)W_t).$$

For a skew diffusion process:

$$X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \sigma(X_s(x))dW_s + (2\alpha - 1)L_t^0(X),$$

a “frozen process" X^y which is the unique strong solution to the equation

$$X_t^y = x + \sigma(y)W_t + (2\alpha - 1)L_t^0(X^y),$$

which is a slightly generalized version of “skew Brownian motion".

The solution of the equation

$$Y_t = x + W_t + (2\alpha - 1)L_t^0(Y),$$

is called the “skew Brownian motion" (Harrison and Shepp [4]).
The density function of Y_t, $p_{Y_t}(x,\cdot)$, can be given explicitly by using the Gaussian density (Walsh, [10]):
if $x \geq 0$

$$p_{Y_t}(x,y) = (g_t(y-x) + (2\alpha - 1)g_t(y+x))\,1(y \geq 0) + 2(1-\alpha)g_t(y-x)\,1(y < 0),$$

and if $x < 0$

$$p_{Y_t}(x,y) = (g_t(y-x) + (1 - 2\alpha)g_t(y+x))\,1(y < 0) + 2\alpha g_t(y-x)\,1(y \geq 0).$$

Note that $p_{Y_t}(x,y)$ satisfies the following condition:

$$\alpha\partial_x p_{Y_t}(0+,y) = (1-\alpha)\partial_x p_{Y_t}(0-,y)$$

and if $\alpha \neq 1/2$, $p_{Y_t}(x,\cdot)$ is discontinuous at 0 because

$$p_{Y_t}(x,0+) = 2\alpha g_t(x) \textit{ and } p_{Y_t}(x,0-) = 2(1-\alpha)g_t(x).$$

In the same way, the density $p_t^y(x,\cdot)$of $X_t^y = x + \sigma(y)W_t + (2\alpha - 1)L_t^0(X^y)$ is given explicitly. We denote $\overline{p}_t(x,y) := p_t^y(x,y)$.
Then using the “semigroup approach", we can prove that

$$p_t(x,y) := \sum_{n=0}^{\infty} \overline{p} \circledast \Phi^{\circledast n}(t,x,y)$$

is the density function of a skew diffusion $X_t(x)$ and a Gaussian upper bound holds. Moreover, $p_t(x,y)$ has the same property of $p_{Y_t}(x,y)$:

$$\alpha\partial_x p_t(0+,y) = (1-\alpha)\partial_x p_t(0-,y).$$

and if $\alpha \neq 1/2$, $p_t(x,\cdot)$ is discontinuous at 0.

Probabilistic representation to use Monte Carlo simulation

Euler-Maruyama scheme

We first note that the Euler-Maruyama scheme for skew diffusion process with out drift term.

Euler-Maruyama scheme:

$$X_t^{(n)} = x + \int_0^t \sigma(X_{\eta_n(s)}^{(n)}) dW_s + (2\alpha - 1) L_t^0(X^{(n)}),$$

where $\eta_n(s) := kT/n$ if $s \in [kT/n, (k+1)T/n)$.

· Assume that σ is bounded, uniformly elliptic and $1/2$-Hölder continuous. Then Using Yamada and Watanabe technique and Le Gall technique, we can prove that there exists a positive constant C such that

$$\mathbb{E}[|X_T - X_T^{(n)}|] \leq \frac{C}{\log n}.$$

Remark 2

Note that this convergence rate is the same one for the Euler-Maruyama scheme for diffusion process. (Gyöngy and Rásonyi [3] or Ngo and Taguchi [9]).

Hence if the diffusion coefficient is Hölder continuous, the E-M scheme may has slow convergence rate.

The density function of $X_t(x)$ has a probabilistic representation which can be used for Monte Carlo simulation.

Theorem 2

Under the same condition as in Theorem 1 for the coefficients, we have the following probabilistic representation: for any $(x, y) \in \mathbb{R}_0 \times \mathbb{R}$*,*

$$p_T(x, y) = \mathbb{E}[H(\tau_1, \cdots, \tau_{R_T}, x, y)],$$

$$H(\tau_1, \cdots, \tau_{R_T}, x, y) := \frac{\overline{p}_{T-\tau_{R_T}}(x, Y^{*,\pi}_{\tau_{R_T}}(y))}{1 - F_\zeta(T - \tau_{R_T})} \prod_{i=0}^{R_T - 1} \frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y), Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1} - \tau_i)},$$

· *for a partition* $\pi_0 = (s_i \wedge T)_{n \in \mathbb{N}}$*,* $Y^{*,\pi_0}(y)$ *is a Markov chain starting at* y *and its transition probability is*

$$\mathbb{P}(Y^{*,\pi_0}_{s_k}(y) \in dy_{k+1} | Y^{*,\pi_0}_{s_{k-1}}(y) = y_k) = \varphi^{y_k}_{s_k - s_{k-1}}(y_{k+1}) dy_{k+1},$$

· $\hat{\theta}_t(x, y) := (L - L^y)\overline{p}_t(\cdot, y)(x)/\varphi^y_t(x)$*,*
· *a counting process* $R_t := \sum_{n=1}^{\infty} 1(\tau_n \le t)$*, interval of jumps* $\tau_n - \tau_{n-1}$ *has the density function* ζ *which is a positive on* $(0, T]$*,*
· *the random partition* $\pi = (\tau_n \wedge T)_{n \in \mathbb{N}}$*, and* $F_\zeta(x) := \int_{-\infty}^{x} \zeta(z) dz$*.*

Therefore, for any function g with $\mathbb{E}[|g(X_T(x))|] < \infty$ and random variable Z with density function f independent from W and R, we have

$$
\begin{aligned}
\mathbb{E}[g(X_T)] &= \int_{\mathbb{R}} g(y)p_T(x,y)dy \\
&= \int_{\mathbb{R}} \frac{g(y)}{f(y)} f(y)p_T(x,y)dy \\
&= \mathbb{E}\left[\frac{g(Z)}{f(Z)} p_T(x,Z)\right] \\
&= \mathbb{E}\left[\frac{g(Z)}{f(Z)} H(\tau_1,\cdots,\tau_{R_T},x,Z)\right].
\end{aligned}
$$

Remark 3

Note that since the density of X_T satisfies a Gaussian upper bound, $\mathbb{E}[g(X_T)] < \infty$ holds for any $|g(x)| \le Ce^{C|x|}$.

Idea of proof

Recall that $p_t(x,y) = \sum_{n=0}^{\infty} \overline{p} \circledast \Phi^{\circledast n}(t,x,y)$. By the definition of convolution $\circledast$, we have

$$\overline{p} \circledast \Phi^{\circledast n}(t,x,y) = \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^n} dy_1 \cdots dy_n \times \prod_{i=0}^{n-1} \Phi_{t_i - t_{i+1}}(y_{i+1}, y_i) \overline{p}_{t_n}(y_{n+1}, y_n).$$

Using the following lemma, we can prove the probabilistic representation for the density of skew diffusion $p_t(x,y)$.

Lemma 1

Let $R = (R_t)_{t \geq 0}$ be a counting process with $((\tau_n)_{n \in \mathbb{N}}, \zeta)$. Then for any $t > 0$, $n \in \mathbb{N}$ and measurable function $H : \mathbb{R}^n \to \mathbb{R}$,

$$\mathbb{E}[1(R_t = n) H(\tau_1, \cdots, \tau_n)]$$

$$= \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 H(s_1, \cdots, s_n)(1 - F_\zeta(t - s_n)) \prod_{i=0}^{n-1} \zeta(s_{i+1} - s_i),$$

where $F_\zeta(x) := \int_{-\infty}^x \zeta(y) dy$ and $s_0 = 0$.

Second moment problem

Let R be the Poisson process with intensity λ, i.e., $\zeta(x) = \lambda e^{-\lambda t}$ Then we have

$$H(\tau_1, \cdots, \tau_{R_T}, x, y) := e^{\lambda T} \lambda^{-R_T} \overline{p}_{T-\tau_{R_T}}(x, Y^{*,\pi}_{\tau_{R_T}}(y)) \prod_{i=0}^{R_T-1} \hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y), Y^{*,\pi}_{\tau_i}(y)).$$

Note that

$$|\hat{\theta}_t(x, y)| \leq \frac{C}{t^{1-\eta/2}} g_t\left(\frac{y-x}{c}\right).$$

Hence we have “formally"

$$\begin{aligned}
&\mathbb{E}[H^2(\tau_1, \cdots, \tau_T, y, x)] \\
&\leq \sum_{n=0}^{\infty} e^{\lambda T} \lambda^{-n} \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \prod_{i=0}^{n-1} \frac{C^2}{(s_{i+1} - s_i)^{2-\eta}} g_{t_0}^2 \left(\frac{y-x}{c}\right) \\
&“=”\infty.
\end{aligned}$$

Finite moment scheme

To reduce the variance of our probabilistic representation, we define the function $\zeta(t) := \frac{A}{t^\beta}\mathbf{1}_{(0,2T]}(t)$ where $A := (1-\beta)/(2T)^{1-\beta}$ and $\beta \in (0,1)$. Then for any $p \geq 2$,

$$\mathbb{E}[H^p(\tau_1, \cdots, \tau_T, y, x)]$$

$$\leq \sum_{n=0}^{\infty} \int_0^{t_0} ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \frac{C_p^n g_{t_0}^p\left(\frac{y-x}{c}\right)}{\prod_{i=0}^{n-1}(s_{i+1}-s_i)^{p-p\eta/2}\zeta^{p-1}(s_{i+1}-s_i)}$$

$$\leq \sum_{n=0}^{\infty} \int_0^{t_0} ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \frac{C_p^n g_{t_0}^p\left(\frac{y-x}{c}\right)}{\prod_{i=0}^{n-1}(s_{i+1}-s_i)^{(p-p\eta/2)-(p-1)\beta}}.$$

By taking $\beta \in (p(1-\eta/2)-1)/(p-1), 1)$, the above series is finite. Note that if β is small,

$$\mathbb{E}[\tau_n - \tau_{n-1}] = \frac{2T(1-\beta)}{2-\beta} \sim T.$$

⇒This implies that by choosing small β, we can control simulation time.

Application to Mathematical Finance

Note that our Finite moment numerical simulation scheme can be useful to compute a “Greeks" in math finance. Indeed, for any $x \in \mathbb{R}_0$, we have

$$\begin{aligned}\partial_x \mathbb{E}[g(X_T(x))] &= \int_{\mathbb{R}} g(y)\partial_x p_T(x,y)dy \\ &= \mathbb{E}\left[\frac{g(Z)}{f(Z)}\partial_x H(\tau_1,\cdots,\tau_{R_T},x,Z)\right],\end{aligned}$$

where $Z \sim f$ and

$$\partial_x H(\tau_1,\cdots,\tau_{R_T},x,y) = \frac{\partial_x \overline{p}_{T-\tau_{R_T}}(x, Y^{*,\pi}_{\tau_{R_T}}(y))}{1-F_\zeta(T-\tau_{R_T})}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y), Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}.$$

Future works

Numerical scheme based on the “parametrix method" for SDE with two reflections:

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \beta_1 L_t^{a_1}(X) + \beta_2 L_t^{a_2}(X)$$

and more generally, for some measure ν on $\mathbb{R}$,

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_{\mathbb{R}} L_t^y(X)\nu(dy).$$

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