Parametrix method for skew diffusion

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A skew diffusion is the unique solution of the following one-dimensional stochastic differential equation with local time:

$$X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \sigma(X_s(x))dW_s + (2\alpha - 1)L_t^0(X), t \in [0,T], \alpha \in (0,1), \quad (1)$$

where W is a one-dimensional standard Brownian motion. The stochastic process $L^0(X)$ is a symmetric local time of X at the origin, that is $L_t^0(X)$ is defined by

$$L_t^0(X) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[-\varepsilon,\varepsilon]}(X_s) d\langle X \rangle_s.$$

The simplest example of skew diffusion process is a skew Brownian motion which is the solution of (1) with $b = 0$ and $\sigma = 1$. Harrison and Shepp [3] proved that if $|2\alpha - 1| \leq 1$ then there is a unique strong solution and if $|2\alpha - 1| > 1$ and $x_0 = 0$, there is no solution. The idea of the proof is a transformation technique to relate (1) with another stochastic differential equation without local time and with discontinuous diffusion coefficient.

In this talk, we prove that the existence and a Gaussian upper bound for the density of a skew diffusion. The idea of proof is the parametrix method for the semigroup $P_t f(x) := \mathbb{E}[f(X_t(x))]$ which is a "Taylor-like expansion". Using the "Backward" parametrix method which is introduced in [2] and [1], we prove the expansion for the semigroup of SDE associated to (1) and its density, under the condition that the drift coefficient is bounded, measurable and the diffusion coefficient is bounded, uniformly elliptic and Hölder continuous. We also obtain the similar expansion for skew diffusion.

In this talk, we also consider a probabilistic representation which can be used Monte Carlo simulation and/or infinite dimensional analysis. More precisely, the parametrix expansion for the density of X_T, $p_T(x, .)$, leads to that for given $p \geq 1$, there exists a random variable $H(T, x, y)$ such that for any $(x, y) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$,

$$p_T(x, y) = \mathbb{E}[H(T, x, y)] \text{ and } \mathbb{E}[|H(T, x, y)|^p] < \infty.$$

References

[1] Bally, V. and Kohatsu-Higa, A. : A probabilistic interpretation of the parametrix method. Ann. Appl. Probab., to appear.

[2] Friedman, A. : Partial Differential Equations of Parabolic Type. Dover Publications, Inc., (1964).

[3] Harrison, J.M. and Shepp, L.A.: On skew Brownian motion. Ann. Probab., 9, 309-313, (1981).

[4] Kohatsu-Higa, A., Taguchi, D. and Zhong, J. Parametrix method for skew diffusions. Preprint, 2015.