Convergence of Brownian motions on RCD spaces

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Motivation

convergence of spaces $\&$ convergence of Brownian motions

- $\mathcal{X}_n = (X_n, d_n, m_n)$ "good" metric measure space
- $\mathrm{Ch_n}(f) = \frac{1}{2}\int_{X_n} |\nabla f|_w^2 dm_n$ Cheeger energy on $\mathcal{X}_n$
- $\mathbb{B}_n = (\{B_t^n\}_{t\geq 0}, \{\mathbb{P}_n^x\}_{x\in X_n})$ Brownian motion on $\mathcal{X}_n$ (ass. w. Ch_n)

$$
\begin{array}{ccc}
\mathcal{X}_n & \xrightarrow{\textbf{(A)}\ \text{mGH}} & \mathcal{X}_\infty \\
{\scriptstyle \mathrm{Ch}_n}\downarrow & & \downarrow{\scriptstyle \mathrm{Ch}_\infty} \\
\mathbb{B}_n & \xrightarrow{\textbf{(B)}\ \text{“in law”}} & \mathbb{B}_\infty
\end{array}
$$

(Q) Does **(A)** imply **(B)** (or, vice versa) ?

Assumption

Assumption

Let $1 < N < \infty,\ K \in \mathbb{R},\ 0 < D < \infty$.

Let $\mathcal{X}_n = (X_n, d_n, m_n)$ be metric measure spaces s.t.

$$\begin{cases} \mathrm{RCD}^*(K,N) & (\text{“Ricci} \geq K,\ \text{dim} \leq N\text{”}) \\ \mathrm{Diam}(X_n) \leq D \\ m_n(X_n) = 1, \end{cases}$$

for $\forall n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ (such $\mathcal{X}_n$ is compact).

Example of RCD^*

- mGH limit of N-dim. compl. Riem. mfd with $\mathrm{Ricci} \geq K \implies \mathrm{RCD}^*(K,N)$.
- N-dim. Alexandrov sp. with $\mathrm{Curv} \geq K \implies \mathrm{RCD}^*((N-1)K, N)$ (Petrunin ’11& Zhang--Zhu ’10).

Main result

Theorem

Under **Assumption**, the following **(A)** and **(B)** are **equivalent**:

(A) $\mathcal{X}_n \stackrel{mGH}{\to} \mathcal{X}_\infty$;

(B) There exist

$$\begin{cases} \text{a compact metric space } (X, d) \\ \text{isometric embeddings } \iota_n : X_n \to X \quad (n \in \overline{\mathbb{N}}) \\ x_n \in X_n \quad (n \in \overline{\mathbb{N}}) \end{cases}$$

such that

$$\iota_n(B^n_\cdot)_\# \mathbb{P}^{x_n}_n \to \iota_\infty(B^\infty_\cdot)_\# \mathbb{P}^{x_\infty}_\infty \quad \text{weakly}$$

in $\mathcal{P}(C([0,\infty); X))$.

$\mathrm{RCD}^*(K, N)$ spaces

L^2-Wasserstein space

- (X, d) complete separable metric space
- $\mu \in \mathcal{P}_2(X, d) \overset{def}{\iff} \mu$: Borel prob. meas. on X s.t.

$$\int_X d^2(x, \overline{x}) d\mu(x) < \infty \quad \text{for some } \overline{x} \in X.$$

- (Coupling) $q \in \Pi(\mu, \nu) \overset{def}{\iff} q$: Borel prob. meas. on $X \times X$ s.t.

$$q(A \times X) = \mu(A), \quad q(X \times A) = \nu(A), \quad \text{(coupl. of } \mu \text{ and } \nu\text{)}.$$

- (L^2-Wasserstein distance) For $\mu, \nu \in \mathcal{P}_2(X, d)$

$$W_2(\mu, \nu) = \left(\inf_{q \in \Pi(\mu,\nu)} \int_{X \times X} d^2(x, y) dq(x, y) \right)^{1/2}.$$

$\mathrm{RCD}^*(K, N)$ spaces

L^2-Wasserstein space

- q optimal coupl. of $\mu, \nu \overset{def}{\iff}$ q attains the following (such q exists)

$$W_2(\mu, \nu) = \Big(\inf_{q \in \Pi(\mu,\nu)} \int_{X \times X} d^2(x, y) dq(x, y) \Big)^{1/2}.$$

- $(\mathcal{P}_2(X, d), W_2)$ complete separable metric space (L^2-Wasserstein space)
- m loc. finite Borel meas. on (X, d).
- $\mu \in \mathcal{P}_\infty(X, d, m) \overset{def}{\iff} \mu \in P_2(X, d)$ & $\mu \ll m$ & bdd support.

$\mathrm{RCD}^*(K, N)$ spaces

volume distortion

- Set, for $\theta \in [0, \infty)$,

$$\Theta_\kappa(\theta) = \begin{cases} \dfrac{\sin(\sqrt{\kappa}\theta)}{\sqrt{\kappa}} & \text{if} \quad \kappa > 0, \\ \theta & \text{if} \quad \kappa = 0, \\ \dfrac{\sinh(\sqrt{-\kappa}\theta)}{\sqrt{-\kappa}} & \text{if} \quad \kappa < 0. \end{cases}$$

- Set, for $t \in [0, 1]$,

$$\sigma_\kappa^{(t)}(\theta) = \begin{cases} \dfrac{\Theta_\kappa(t\theta)}{\Theta_\kappa(\theta)} & \text{if} \quad \kappa\theta^2 \neq 0 \quad \text{and} \quad \kappa\theta^2 < \pi^2, \\ t & \text{if} \quad \kappa\theta^2 = 0, \\ +\infty & \text{if} \quad \kappa\theta^2 \geq \pi^2. \end{cases}$$

$\mathrm{RCD}^*(K, N)$ spaces

$\mathrm{CD}^*(K, N)$

Definition (Bacher–Sturm '10) Let $K \in \mathbb{R}$ and $1 < N < \infty$.

(X, d, m) satisfies $\mathrm{CD}^*(K, N)$ $\overset{def}{\Longleftrightarrow}$ $\forall \mu_0, \mu_1 \in \mathcal{P}_\infty(X, d, m)$,

$$\begin{cases} \exists \text{ opt. coupl. } q \text{ of } \mu_0 \text{ and } \mu_1 \\ \exists \text{geod. } \mu_t = \rho_t m \in \mathcal{P}_\infty(X, d, m) \text{ connect. } \mu_0 \text{ and } \mu_1 \end{cases}$$

s.t.

$$\int \rho_t^{-\frac{1}{N'}} d\mu_t \geq \int_{X \times X} \Big[\sigma_{K/N'}^{(1-t)}(d(x_0, x_1)) \rho_0^{-1/N'}(x_0) + \sigma_{K/N'}^{(t)}(d(x_0, x_1)) \rho_1^{-1/N'}(x_1) \Big] dq(x_0, x_1),$$

for $\forall t \in [0, 1]$ and $\forall N' \geq N$.

$\mathrm{RCD}^*(K, N)$ spaces

Sturm'06, Bacher--Sturm'10, Erbar--Kuwada--Sturm'15

When $(X, d, m) = (M, d_g, m_g)$ N-dim. compl. Riem. mfd.

$$\mathrm{Ricci} \geq K \iff \mathrm{CD}^*(K, N)$$

$$\mathrm{Ric}_x \geq K g_x \iff \int \rho_t^{-\frac{1}{N'}} d\mu_t \geq \int_{X \times X} \Big[\sigma_{K/N'}^{(1-t)}(d(x_0, x_1)) \rho_0^{-1/N'}(x_0) + \sigma_{K/N'}^{(t)}(d(x_0, x_1)) \rho_1^{-1/N'}(x_1) \Big] dq(x_0, x_1).$$

Functional inequalities, volume growth...

Poincaré ineq. Sobolev ineq. Bishop--Gromov ineq. Bonnet--Myers.

$\mathrm{RCD}^*(K, N)$ spaces

Cheeger energy

- For $f \in \mathrm{Lip}(X)$, local Lip. const. $|\nabla f| \overset{def}{\Longleftrightarrow}$

$$|\nabla f|(x) = \begin{cases} \limsup_{y \to x} \dfrac{|f(y) - f(x)|}{d(y, x)} & \text{if } x \text{ is not isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

- Ch : $W^{1,2}(X, d, m) \to \mathbb{R}$: Cheeger energy $\overset{def}{\Longleftrightarrow}$

$$\mathrm{Ch}(f) = \frac{1}{2} \inf\Big\{ \liminf_{n\to\infty} \int |\nabla f_n|^2 dm : f_n \in \mathrm{Lip}(X), \int_X |f_n - f|^2 dm \to 0 \Big\}$$

$$f \in W^{1,2}(X, d, m) = \{f \in L^2(X, m) : \mathrm{Ch}(f) < \infty\}.$$

- Ch is **NOT** necessarily **quadratic**.

$\mathrm{RCD}^*(K, N)$ spaces

$\mathrm{RCD}^*(K, N)$

Definition (Erbar--Kuwada--Sturm (*to appear in Invent. math.*))

(X, d, m) satisfies $\mathrm{RCD}^*(K, N) \overset{def}{\Longleftrightarrow}$

$$\begin{cases} \mathrm{CD}^*(K, N) \\ \text{Cheeger energy Ch is quadratic:} \end{cases}$$

$$2\mathrm{Ch}(f) + 2\mathrm{Ch}(g) = \mathrm{Ch}(f+g) + \mathrm{Ch}(f-g), \quad \forall f, g \in W^{1,2}(X, d, m).$$

Assumption (again)

Assumption

Let $1 < N < \infty,\ K \in \mathbb{R},\ 0 < D < \infty$.

Let $\mathcal{X}_n = (X_n, d_n, m_n)$ be metric measure spaces s.t.

$$\begin{cases} \mathrm{RCD}^*(K, N) \\ \mathsf{Diam}(X_n) \le D \\ m_n(X_n) = 1, \end{cases}$$

for $\forall n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

Brownian motions on $\mathrm{RCD}^*(K, N)$ spaces

Brownian motion

Under **Assumption**,

- strongly local regular conservative Dirichlet form

$$\mathcal{E}_n(u, v) := \frac{1}{4}\big(\mathsf{Ch}_n(u+v) - \mathsf{Ch}_n(u-v)\big),$$
$$D[\mathcal{E}_n] := W^{1,2}(X_n, d_n, m_n).$$

- $\{T_t^n\}_{t \geq 0}$ strong Feller semigroup, i.e.

$$\begin{cases} T_t^n f \in C_\infty(X_n) & (\forall f \in \mathcal{B}_b(X_n) \cap L^2(X_n, m_n), \forall t > 0), \\ \lim_{t \downarrow 0} \|T_t^n f - f\|_\infty = 0 & (\forall f \in C_\infty(X_n)). \end{cases}$$

Brownian motions on $\mathrm{RCD}^*(K, N)$ spaces

Brownian motion

- (Brownian motion) $\exists$ conti. Markov process $(\{B_t^n\}_{t\geq 0}, \{\mathbb{P}_n^x\}_{x\in X_n})$ s.t.

$$T_t^n f(x) = \mathbb{E}_n^x(f(B_t^n)) := \int_{\Omega_n} f(B_t^n)\, d\mathbb{P}_n^x \qquad (\forall t > 0,\ \forall x \in X_n),$$

for $\forall f \in \mathcal{B}_b(X_n) \cap L^2(X_n, m_n)$.

mGH

measured Gromov--Hausdorff conv.

- (X_n, d_n, m_n) compact metric measure spaces.
- **measured Gromov-Hausdorff conv. (Fukaya '87)**

  $(X_n, d_n, m_n) \stackrel{mGH}{\to} (X_\infty, d_\infty, m_\infty) \stackrel{def}{\iff} \exists \varepsilon_n \downarrow 0$, and $f_n : X_n \to X_\infty$ Borel, s.t.

  (i) **(almost isometry)** $\sup_{x,y \in X_n} |d_n(x,y) - d_\infty(f_n(x), f_n(y))| \leq \varepsilon_n$;

  (ii) **(almost surjective)** $X_\infty \subset B_{\varepsilon_n}(f_n(X_n))$;

  (iii) **(convergence of measures)** for any $\phi \in C_b(X_\infty)$,

$$\lim_{n\to\infty} \int_{X_n} \phi \circ f_n \, dm_n \to \int_{X_\infty} \phi \, dm_\infty.$$

Main result (again)

Theorem

Under **Assumption**, the following **(A)** and **(B)** are **equivalent**:

(A) $\mathcal{X}_n \overset{mGH}{\to} \mathcal{X}_\infty$;

(B) There exist

$$\begin{cases} \text{a compact metric space } (X, d) \\ \text{isometric embeddings } \iota_n : X_n \to X \quad (n \in \overline{\mathbb{N}}) \\ x_n \in X_n \quad (n \in \overline{\mathbb{N}}) \end{cases}$$

such that

$$\iota_n(B^n_\cdot)_\# \mathbb{P}^{x_n}_n \to \iota_\infty(B^\infty_\cdot)_\# \mathbb{P}^{x_\infty}_\infty \quad \text{weakly}$$

in $\mathcal{P}(C([0,\infty); X))$.

Related result

Theorem (Ogura '01)

Assume the following two assumptions hold,

- M_n compact connect. weighted Riem. mfd (M_n, g_n, w_n) s.t. $\exists \alpha,\ \nu,\ \beta > 0$ indep. n s.t.

$$p_n(t,x,y) \le \frac{\alpha}{(t \wedge 1)^{\nu/2}} \quad t > 0,\ x, y \in M,$$
$$\mu_g^w(M_n) \le \beta,$$

- $\exists (X_\infty, d_\infty, m_\infty), \exists$ cont. heat ker. p_∞ s.t. $M_n \overset{Kasue-Kumura}{\longrightarrow} X_\infty$ with ε_n-isom. $f_n : M_n \to X_\infty$, and $f_n(x_n) \to x_\infty \in X_\infty$.

Then **after time-discretization** $\phi = \phi_{\varepsilon_n}$

$$f_n(B^n_{\phi(\cdot)})_{\#} P_n^{x_n} \overset{weak}{\longrightarrow} B^\infty_{\phi(\cdot)}{}_{\#} P_\infty^{x_\infty}$$

in $\mathcal{P}(D([0,\infty); X_\infty))$.

Sketch: (A) $\Longrightarrow$ (B)

embedding

Fact

Under **Assumption**, the following (A) and (A′) are equivalent:

(A) $\mathcal{X}_n \overset{mGH}{\to} \mathcal{X}_\infty$

(A′) There exist

$$\begin{cases} \text{a compact metric space } (X, d) \\ \text{isometric embeddings } \iota_n : X_n \to X \quad (n \in \overline{\mathbb{N}}) \end{cases}$$

such that

$$\iota_{n\#} m_n \to \iota_{\infty\#} m_\infty \quad \text{weakly} \quad \text{in} \quad \mathcal{P}(X).$$

- (A′) implies $\iota_n(X_n) \overset{Hausdorff}{\to} \iota_\infty(X_\infty)$ in (X, d).

Sketch: (A) $\Longrightarrow$ (B)

plan

- Identify $\iota_n(X_n) \sim X_n$.
- Let $x_n \in X_n$ s.t.
$$\iota_n(x_n) \to \iota_\infty(x_\infty).$$
- $\mathbb{B}_n := \iota_n(B^n_\cdot)_\# \mathbb{P}^{x_n}_n$ for short.

We show

- $\{\mathbb{B}_n\}_{n\in\mathbb{N}}$ relatively compact in $\mathcal{P}(C([0,\infty);X))$
- Uniqueness of limit

Sketch: (A) $\Longrightarrow$ (B)

relative compactness

- (moment estimate) $\forall T > 0, \exists \beta > 0, \exists C > 0, \exists \theta > 1$ s.t. $\forall n \in \mathbb{N}$

$$\mathbb{E}^{x_n}[d^\beta(B_t^n, B_{t+h}^n)] \le Ch^\theta, \quad (0 \le t \le T, \quad 0 \le h \le 1). \tag{1}$$

- (unif. doubling) + (unif. weak $(1,2)$-Poincaré) + (Bishop--Gromov ineq.)

  $\Longrightarrow$ uniform Gaussian heat kernel estimate:

  $\exists C_1, C_2, \nu > 0$ indep. of n s.t.

$$p(t,x,y) \le \frac{C_1}{t^\nu} \exp\left\{-C_2 \frac{d(x,y)^2}{t}\right\}, \tag{2}$$

  for all $x, y \in X$ and $0 < t \le 1 \wedge D^2$.
- (2) $\Longrightarrow$ (1).

Sketch: (A) $\Longrightarrow$ (B)

uniqueness

- (Sufficient cond.) For all $k \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \infty$, $g_1, g_2, \ldots, g_k \in C_b(X)$,

$$\mathbb{E}^{x_n}[g_1(B^n_{t_1}) \cdots g_k(B^n_{t_k})] \to \mathbb{E}^{x_\infty}[g_1(B^\infty_{t_1}) \cdots g_k(B^\infty_{t_\infty})]. \tag{3}$$

(Convergence of finite-dimensional distributions).

- Recall

$$T^n_t f(x_n) = \mathbb{E}^x_n(f(B^n_t)).$$

- We would like to show

$$T^n_t f \overset{\text{“uniform”}}{\to} T^\infty_t f \quad \text{on} \quad X$$

for any $f \in C_b(X)$, $t > 0$.

Sketch: (A) $\Longrightarrow$ (B)

extention

- We would like to show

$$T_t^n f \overset{\text{“uniform”}}{\to} T_t^\infty f \quad \text{on} \quad X$$

  for any $f \in C_b(X)$, $t > 0$.

- $T_t^n f$ is NOT CONTINUOUS (not defined) on the whole space X.

- (By unif. Parabolic Harnack ineq.) $T_t^n f$ unif. Hölder on X_n:
  $\exists\, 0 < \alpha < 1, 0 < H < \infty$ indep. of n s.t.

$$|T_t^n f(x) - T_t^n f(y)| \leq H d_n(x,y)^\alpha.$$

- Extend to the whole space X

$$\widetilde{T_t^n f}(x) := \sup_{a \in X_n}\{T_t^n f(a) - H d(a,x)^\alpha\} \quad x \in X. \tag{4}$$

Sketch: (A) $\implies$ (B)

extention

Then we have

- $\widetilde{T_t^n f} = T_t^n f$ on X_n.
- $\widetilde{T_t^n f}$ is (α, H)-Hölder on the whole space X.

Now we would like to show

$$\widetilde{T_t^n f} \stackrel{uniform}{\to} \widetilde{T_t^\infty f} \quad \text{on} \quad X.$$

Sketch: (A) $\implies$ (B)

uniqueness

Now we would like to show

$$\widetilde{T_t^n f} \stackrel{uniform}{\to} \widetilde{T_t^\infty f} \quad \text{on} \quad X.$$

- (equi-continuity) + (Mosco conv. of Ch in (Gigli–Mondino–Savaré '13))

  $\implies$ If $\{\widetilde{T_t^n f}\}_n$ has two converging subseq. to F_1 and F_2, then

$$F_1|_{X_\infty} = F_2|_{X_\infty} = T_t^\infty f.$$

- It suffices to show

$$\widetilde{F_1|_{X_\infty}} = F_1.$$

- Key: stability of extention $\sim$ under Hausdorff conv. $X_n \to X_\infty$

Finish $(A) \Rightarrow (B)$

Sketch: (B) $\Longrightarrow$ (A)

Ergodic theorem

(Q) How to get information of the underlying space from B.M. ?

(A) Ergodic theorem

- B.M. recurrent & irreducible & $\exists p(t, x, y)$
- (Ergodic theorem) For any $f \in L^1(X_n, m_n)$,

$$\mathbb{E}_n^x(f(B_t^n)) \overset{t\to\infty}{\to} \frac{1}{m_n(X_n)} \int_{X_n} f \, dm_n.$$

Sketch: (B) $\Longrightarrow$ (A)

Ergodic theorem

- For open $G \subset X$,

$$\liminf_{n\to\infty} m_n(G) \overset{\text{Ergodic}}{=} \liminf_{n\to\infty} \lim_{t\to\infty} \mathbb{P}_n^{x_n}(B_t^n \in G) \overset{?}{=} \lim_{t\to\infty} \liminf_{n\to\infty} \mathbb{P}_n^{x_n}(B_t^n \in G)$$

$$\overset{(B)}{\geq} \lim_{t\to\infty} \mathbb{P}_\infty^{x_\infty}(B_t^\infty \in G) \overset{\text{Ergodic}}{=} m_\infty(G).$$

- It suffices to show

$$\liminf_{n\to\infty} \lim_{t\to\infty} \mathbb{P}_n^{x_n}(B_t^n \in G) = \lim_{t\to\infty} \liminf_{n\to\infty} \mathbb{P}_n^{x_n}(B_t^n \in G).$$

- Key: Speed of convergence to equillibrium as $t \to \infty$ need to be controlled independently of n.

Sketch: (B) $\Longrightarrow$ (A)

Spectral gap

- Let λ_n^1 be the spectral gap of Ch$_n$:

$$\lambda_n^1 := \inf\{\frac{\mathsf{Ch}_n(f)}{\|f\|^2_{L^2(m_n)}} : f \in \mathrm{Lip}(X_n) \setminus \{0\}, \int_{X_n} f dm_n = 0\}. \tag{5}$$

- Speed to equilibrium are controlled by λ_n^1:

$$\|T_t^n f - m_n(f)\|_{L^2(m_n)} \leq e^{-\lambda_n^1 t}\|f - m_n(f)\|_{L^2(m_n)}, \tag{6}$$

  for any $t > 0$. Here $m_n(f) := \int_{X_n} f dm_n$.
- Under **assumption** (unif. Poincaré ineq.), we have

$$\inf_{n\in\mathbb{N}} \lambda_n^1 > 0.$$

Finish the proof & Thank you.

Future works

- How is $\mathrm{RCD}(K,\infty)$? (in progress)
- How is $\mathrm{RCD}^*(K,N)$ **WITHOUT diameter bounds**?
- How is discrete cases?

Discrete ver. of "Ricci":

$$\Longrightarrow \begin{cases} \text{Coarse Ricci curvature (Ollivier '09),} \\ \text{Rough CD (Bonciocat--Sturm '09, Bonciocat '14)} \end{cases}$$

Thank you.