Convergence of Brownian motions on RCD spaces

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1 Introduction & Result

In this talk, we consider the following problem:

(Q) Does **the weak convergence** of Brownian motions follow **only** from **geometrical convergence** of the underlying spaces (or, vice versa)?

As a main result in this talk, we show that the weak convergence of the laws of Brownian motions is **equivalent** to the measured Gromov–Hausdorff (mGH) convergence of the underlying metric measure spaces under the following assumption:

Assumption 1.1 Let N, K and D be constants with $1 < N < \infty$, $K \in \mathbb{R}$ and $0 < D < \infty$. For $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, let $\mathcal{X}_n = (X_n, d_n, m_n)$ be a metric measure space satisfying the $\mathrm{RCD}^*(K, N)$ condition with $\mathrm{Diam}(X_n) \leq D$ and $m_n(X_n) = 1$.

Under Assumption 1.1, it is known that there exists a conservative Hunt process on $\mathcal{X}_n$ associated with the Cheeger energy and unique in all starting points in $\mathcal{X}_n$. We denote it by $(\{\mathbb{P}_n^x\}_{x \in X_n}, \{B_t^n\}_{t \geq 0})$, called *the Brownian motion on* $\mathcal{X}_n$. We state our main theorem precisely:

Theorem 1.2 *Suppose that Assumption 1.1 holds. Then the following statements (i) and (ii) are equivalent:*

(i) (mGH-convergence of the underlying spaces)

$\mathcal{X}_n$ *converges to* $\mathcal{X}_\infty$ *in the measured Gromov–Hausdorff sense.*

(ii) (Weak convergence of the laws of Brownian motions)

There exist

$$\begin{cases} \text{a compact metric space } (X, d) \\ \text{isometric embeddings } \iota_n : X_n \to X \ (n \in \overline{\mathbb{N}}) \\ x_n \in X_n \ (n \in \overline{\mathbb{N}}) \end{cases}$$

such that

$$\iota_n(B_\cdot^n)_\# \mathbb{P}_n^{x_n} \to \iota_\infty(B_\cdot^\infty)_\# \mathbb{P}_\infty^{x_\infty} \quad \text{weakly} \quad \text{in } \mathcal{P}(C([0,\infty); X)).$$

The subscript $\#$ means the operation of the push-forward of measures.

$\mathbf{RCD}^*(\mathbf{K}, \mathbf{N})$ (Riemannian Curvature-Dimension) spaces, introduced by Erbar–Kuwada–Sturm [2], are metric measure spaces satisfying a generalized notion of "$\mathbf{Ricci} \geq \mathbf{K}, \mathbf{dim} \leq \mathbf{N}$", which include several important classes of non-smooth spaces. For example, measured Gromov–Hausdorff (**mGH**) **limit spaces** of complete Riemannian manifolds with Ricci $\geq K, \dim = N$, or **Alexandrov spaces** with Curv $\geq K/(N-1), \dim = N$ are included in $\mathrm{RCD}^*(K, N)$ spaces.

Remark 1.3 We give comments to several related works.

(i) In [4], Ogura studied the weak convergence of the laws of the Brownian motions on Riemannian manifolds *by a different approach from this talk.* He push-forwarded all Brownian motions not to the ambient space X, but to *the limit space* M_∞ with respect to approximation maps $f_n : M_n \to M_\infty$ of the Kasue–Kumura convergence with certain *time-discretization* of Brownian motions.

More precisely, he assumed uniform upper bounds for heat kernels, and the Kasue–Kumura spectral convergence ([3]) of the underlying manifolds M_n. He push-forward each Brownian motions on M_n to the Kasue–Kumura spectral limit space M_∞ with respect to ε_n-isometry $f_n : M_n \to M_\infty$, and show the convergence in law on the càdlàg space of the push-forwarded and time-discretized Brownian motions on M_∞.

(ii) In [1], Albeverio and Kusuoka studied diffusion processes associated with SDEs on thin tubes in $\mathbb{R}^d$ shrinking to one-dimensional spider graphs. They studied the weak convergence of these diffusions to one-dimensional diffusions on the limit graphs. Their setting does not satisfy the $\mathrm{RCD}^*(K, N)$ condition because Ricci curvatures are not bounded below at points of conjunctions in spider graphs.

References

[1] Albeverio, S. and Kusuoka, S. (2012). Diffusion processes in thin tubes and theier limits on graphs. *Ann. Probab.* **40(5)**. 2131–2167.

[2] Erbar, M., Kuwada, K. and Sturm, K.-Th. On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. *to appear in Invent. Math. Preprint Available at*: arXiv:1303.4382v2.

[3] Kasue, A. and Kumura, H. (1994). Spectral convergence of Riemannian manifolds. *Tôhoku Math. J.* **46** 147–179.

[4] Ogura, Y. (2001). Weak convergence of laws of stochastic processes on Riemannian manifolds. *Probab. Theory Related Fields* **119** 529–557.