On density function concerning discrete time maximum of some one-dimensional diffusion processes

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Introduction

In this talk, we will show some results on the density functions related to discrete time maximum of some one-dimensional diffusion processes.
That is defined by $M_T^n = \max\{X_{t_1}, \cdots, X_{t_n}\}$ for a fixed time interval $[0, T]$ and a time partition $\Delta_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < t_{n+1} = T$ for $n \geq 2$, where $\{X_t, t \in [0, \infty)\}$ denotes a one-dimensional diffusion process.
In particular, we will study on the density function of (M_T^n, X_T) and M_T^n for some one-dimensional diffusions.

Introduction

- The first goal is to prove an integration by parts (IBP) formula for (M_T^n, X_T) in the case that X satisfies an SDE;

$$E[\partial_\beta \varphi(M_T^n, X_T)G] = E[\varphi(M_T^n, X_T)H_\beta(H, G)]$$

  for arbitrary $\varphi \in C_b^\infty(\mathbb{R}^2; \mathbb{R})$.
- The second goal is to obtain asymptotic behaviors of density function of M_T^n and (M_T^n, X_T) for some Gaussian processes.

M_T^n and (M_T^n, X_T) are important especially in finance. Indeed, for "payoff functions" f,

- $f(M_T^n)$: Lookback type option
- $f(M_T^n, X_T)$: Barrier type option

Outline of the talk

1. Previous works
2. IBP formula for (M_T^n, X_T)
3. Asymptotic behavior of density functions of M_T^n and (M_T^n, X_T)

Outline

1. Previous works
2. IBP formula for (M_T^n, X_T)
3. Asymptotic behavior of density functions of M_T^n and (M_T^n, X_T)

Previous works

Study of density functions on maxima.

- Nualart, D., Vives, J.: Continuité absolue de la loi du maximum d'un processus continu. C. R. Acad. Sci. Paris Ser. 1 Math. **307**(7), 349-354 (1988). → A sufficient condition so that the law of continuous time maximum of a one-dimensional continuous process is absolutely continuous.
- Florit, C., Nualart, D.: A local criterion for smoothness of densities and application to the supremum of the Brownian sheet. Stat. Probab. Lett. **22**(1), 25-31 (1995). → The smoothness of the density function of the continuous time maximum of the Brownian sheet.
- Lanjri Zadi, N., Nualart, D.: Smoothness of the law of the supremum of the fractional Brownian motion. Electron. Comm. Probab. **8**, 102-111 (2003). → The smoothness of the density function of the continuous time maximum of the fractional Brownian motion.

Previous works

Study of density functions on maxima.

- Fournier, N., Printems, J.: Absolute continuity for some one-dimensional processes. Bernoulli **16**(2), 343-360 (2010).→Absolute continuity of the law of a solution to a one-dimensional SDE with coefficients depending on the continuous time maximum of the solution.
- Hayashi, M., Kohatsu-Higa, A.: Smoothness of the distribution of the supremum of a multi-dimensional diffusion process. Potential Anal. **38**(1), 57-77 (2013).→ The smoothness of the density function of the joint law of a multi-dimensional SDE at the time when a component attains its continuous time maximum by means of the IBP formula.
- N.: Absolute continuity of the laws of a multi-dimensional stochastic differential equation with coefficients dependent on the maximum. Stat. Probab. Lett. **83**(11), 2499-2506 (2013).→ Multi-dimensional case of [Fournier, N., Printems, J.].
- N.: Integration by parts formulas concerning maxima of some SDEs with applications to study on density functions (preprint).→A little bit of generalization of [Hayashi, M., Kohatsu-Higa, A.] in the case of one-dimension.

Outline

IBP for (M_T^n, X_T)

We consider the one-dimensional SDE:

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

where $b, \sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions and $\{W_t, t \in [0, \infty)\}$ is a one-dimensional standard Brownian motion.
Fix $T > 0$ and a time partition $0 < t_1 < \cdots < t_n = T$ and define $M_T^n := \max\{X_{t_1}, \cdots, X_{t_n}\}$.

IBP for (M^n_T, X_T)

Assumption (A)

(A1) For $t \in [0,\infty)$, $b(t,\cdot), \sigma(t,\cdot) \in C_b^\infty(\mathbb{R};\mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t,\cdot)$ and $\sigma(t,\cdot)$ do not depend on t. In particular, let $c(\sigma)$ be a constant which bounds $|\sigma(t,x)|$.

(A2) There exists $c > 0$ such that

$$|\sigma(t,x)| \geq c$$

holds, for any $x \in \mathbb{R}$ and $t \in [0,\infty)$.

IBP for (M_T^n, X_T)

Theorem 1

Assume (A). Let $G \in \mathbb{D}^\infty$. Then, for any multi index $\beta \in \{1,2\}^k$, $k \geq 1$, there exists $H_\beta(G) \in \mathbb{D}^\infty$ such that

$$E^P[\partial_\beta \varphi(M_T^n, X_T)G] = E^P[\varphi(M_T^n, X_T)H_\beta(G)] \qquad (1)$$

holds for arbitrary $\varphi \in C_b^\infty(\mathbb{R}^2; \mathbb{R})$.

Proof

Define $Y_t := \frac{\partial X_t}{\partial x_0}$ for $t \in [0, T]$ and $A_1 := \{X_{t_1} = M^n_T\}$, $A_k := \{X_{t_1} \neq M^n_T, \cdots, X_{t_{k-1}} \neq M^n_T, X_{t_k} = M^n_T\}$ for $2 \leq k \leq n$. Then, due to the local property of the Mallivin derivative (Proposition 1.3.16 of [16]), we have

$$DM^n_T = \sum_{i=1}^{n} DX_{t_i} \mathbf{1}_{A_i}.$$

Let us consider two processes $\{u^1_r, r \in [0, T]\}$ and $\{u^2_r, r \in [0, T]\}$ defined by

$$u^1_r = \frac{Y_r}{\sigma(r, X_r)} \left[\frac{1}{t_1 Y_{t_1}} \mathbf{1}_{[0,t_1)}(r) + \sum_{k=2}^{n} \left(\frac{1}{Y_{t_k}} - \frac{1}{Y_{t_{k-1}}} \right) \frac{1}{t_k - t_{k-1}} \mathbf{1}_{[t_{k-1}, t_k)}(r) + \frac{1}{T - t_n} \frac{1}{Y_T} \mathbf{1}_{[t_n, T]}(r) \right],$$

$$u^2_r = \frac{Y_r}{\sigma(r, X_r)} \left[\frac{1}{t_1 Y_{t_1}} \mathbf{1}_{[0,t_1)}(r) + \sum_{k=2}^{n} \left(\frac{1}{Y_{t_k}} - \frac{1}{Y_{t_{k-1}}} \right) \frac{1}{t_k - t_{k-1}} \mathbf{1}_{[t_{k-1}, t_k)}(r) + \left(\frac{1}{Y_T} - \frac{1}{Y_{t_n}} \right) \frac{1}{T - t_n} \mathbf{1}_{[t_n, T]}(r) \right],$$

for $r \in [0, T]$.

Proof

Then, by the definition of u^1 and u^2, we easily get

$$\int_0^T D_r M_T^n \cdot u_r^1 dr = \sum_{i=1}^n \mathbf{1}_{A_i} = 1, \int_0^T D_r M_T^n \cdot u_r^2 dr = 1$$

$$\int_0^T D_r X_T \cdot u_r^1 dr = 1 + \frac{Y_T}{Y_{t_n}}, \int_0^T D_r X_T \cdot u_r^2 dr = 1,$$

thus, the determinant of the matrix given by

$$\begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{bmatrix} := \begin{bmatrix} \langle DM_T^n, u^1 \rangle_{L^2([0,T])} & \langle DM_T^n, u^2 \rangle_{L^2([0,T])} \\ \langle DX_T, u^1 \rangle_{L^2([0,T])} & \langle DX_T, u^2 \rangle_{L^2([0,T])} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 + \frac{Y_T}{Y_{t_n}} & 1 \end{bmatrix}$$

is equal to $-Y_T / Y_{t_n}$.

Proof

From (A), one has that $-Y_{t_n}/Y_T \in L^p(\Omega)$ for any $p \geq 1$, $u^i \in \mathbb{D}^\infty(L^2([0,T]))$ for $i = 1, 2$ and $\gamma_{i,j} \in \mathbb{D}^\infty$ for $i, j = 1, 2$. Therefore, we may conclude that (M_T^n, X_T) is a locally nondegenerate random vector in $\mathbb{R}^2$, in the sense of Definition 2.1.2 of [16]. Moreover, one has

$$
\begin{aligned}
E^P[G\partial_\beta\varphi(M_T^n, X_T)] &= E^P\left[\left\langle D(\varphi(M_T^n, X_T)), G\sum_{j=1}^{2}(\gamma^{-1})_{j,\beta}u^j\right\rangle_{L^2([0,T])}\right] \\
&= E^P\left[\varphi(M_T^n, X_T)\delta\left(G\sum_{j=1}^{2}(\gamma^{-1})_{j,\beta}u^j_\cdot\right)\right],
\end{aligned}
$$

for $\beta \in \{1, 2\}$.

Proof

More generally, by defining $H_i(G) = \delta(G \sum_{j=1}^{2} (\gamma^{-1})_{j,i} u_\cdot^j)$, $i \in \{1,2\}$, we obtain

$$E^P\left[G \frac{\partial^k \varphi}{\partial x_{\beta_1} \cdots \partial x_{\beta_{k-1}} \partial x_{\beta_k}}(M_T^n, X_T)\right] = E^P\left[\frac{\partial^{k-1} \varphi}{\partial x_{\beta_1} \cdots \partial x_{\beta_{k-1}}}(M_T^n, X_T) H_{\beta_k}(G)\right]$$

$$\vdots$$

$$= E^P\left[\varphi(M_T^n, X_T) H_{\beta_1}(\cdots H_{\beta_{k-1}}(H_{\beta_k}(G)) \cdots)\right]$$

for $\beta_1, \cdots, \beta_k \in \{1,2\}$. Therefore, (1) holds with

$H_\beta(G) = H_{\beta_1}(\cdots H_{\beta_{k-1}}(H_{\beta_k}(G)) \cdots)$ for $\beta \in \{1,2\}^k$. □

IBP for (M^n_T, X_T)

Corollary 2

The density function of (M^n_T, X_T) *on* $\mathbb{R}^2$ *belongs to* $C_b^\infty(\mathbb{R}^2;\mathbb{R})$ *and is represented as*

$$p_{M^n_T,X_T}(x,y) = E^P\left[\mathbf{1}_{\{M^n_T \geq x\}}\mathbf{1}_{\{X_T \geq y\}} H_{(1,2)}(1)\right].$$

Furthermore, one has

$$\partial_\beta p_{M^n_T,X_T}(x,y) = (-1)^{|\beta|} E^P\left[\mathbf{1}_{\{M^n_T \geq x\}}\mathbf{1}_{\{X_T \geq y\}} H_\beta(H_{(1,2)}(1))\right],$$

for $\beta \in \{1,2\}^k$, $k \in \mathbb{N}$.
Moreover, since $E^P[H_\beta(G)] = 0$ *for* $G \in \mathbb{D}^\infty$, *we easily obtain*

$$p_{M^n_T,X_T}(x,y) = -E^P\left[\mathbf{1}_{\{M^n_T < x\}\cup\{X_T < x\}} H_{(1,2)}(1)\right]$$

and

$$\partial_\beta p_{M^n_T,X_T}(x,y) = (-1)^{|\beta|+1} E^P\left[\mathbf{1}_{\{M^n_T < x\}\cup\{X_T < x\}} H_\beta(H_{(1,2)}(1))\right],$$

for $\beta \in \{1,2\}^k$, $k \in \mathbb{N}$.

IBP for (M_T^n, X_T)

Theorem 3

Assume (A). Then, for $p_1, p_2 > 1$ *and* $\beta \in \{1,2\}^k$, $k \in \mathbb{N} \cup \{0\}$, *there exists* $C(p_1, p_2, \Delta_n, T, \beta) > 0$ *such that*

$$\partial_\beta p_{M_T^n, X_T}(x,y) \leq \frac{C(p_1, p_2, \Delta_n, T, \beta) e^{-\frac{1}{2p_1p_2}\frac{(x\vee y - x_0)^2}{c(\sigma)^2 T}}}{\left((x \vee y - x_0) + \sqrt{(x\vee y - x_0)^2 + 2c(\sigma)^2 T}\right)^{\frac{1}{p_1p_2}}}, x \vee y > x_0$$

and

$$\partial_\beta p_{M_T^n, X_T}(x,y) \leq C(p_1, p_2, \Delta_n, T, \beta) \left(\frac{e^{-\frac{1}{2p_1p_2}\frac{(x-x_0)^2}{c(\sigma)^2 t_1}}}{\left((x_0 - x) + \sqrt{(x_0 - x)^2 + 2c(\sigma)^2 t_1}\right)^{\frac{1}{p_1p_2}}} + \frac{e^{-\frac{1}{2p_1p_2}\frac{(y-x_0)^2}{c(\sigma)^2 T}}}{\left((x_0 - y) + \sqrt{(x_0 - y)^2 + 2c(\sigma)^2 T}\right)^{\frac{1}{p_1p_2}}} \right), x, y < x_0,$$

hold, where we have defined $\partial_\beta p_{M_T^n, X_T} \equiv p_{M_T^n, X_T}$ *for* $k = 0$.

Outline

1. Previous works
2. IBP formula for (M_T^n, X_T)
3. Asymptotic behavior of density functions of M_T^n and (M_T^n, X_T)

Asymptotic behavior

Let us deal with the multiple integral,

$$I(\theta) = \int_R f(x_1, \cdots, x_n) e^{-\theta^2 \phi(x_1, \cdots, x_n) + k(\theta)\psi(x_1, \cdots, x_n)} dx_1 \cdots dx_n,$$

where $R = \prod_{i=1}^n (-\infty, d_i]$ and f, ϕ, ψ are measurable functions defined on $\mathbb{R}^n$ and consider the asymptotic behavior of $I(\theta)$ as $\theta \to \infty$.
We use the notation,

$$f(\theta) \sim g(\theta) \overset{\text{def}}{\iff} \frac{f(\theta)}{g(\theta)} \to 1, (\theta \to \infty).$$

Asymptotic behavior

We assume the following,
Assumption (B)

(B1) $\phi \in C^2(\mathbb{R}^n;\mathbb{R})$ and ϕ attains its global minimum at a unique point $x^* = (x_1^*, \cdots, x_n^*) \in R$, in particular, we assume that $x_{j_1}^* = d_{j_1}, \cdots, x_{j_m}^* = d_{j_m}$ for $1 \le j_1 < \cdots < j_m \le n$, $0 \le m \le n$ and $x_i^* < d_i$ for other $1 \le i \le n$.

(B2) There exist $a_i > 0$ and $b_i \in \mathbb{R}$, $1 \le i \le n$ such that $\phi(x_1, \cdots, x_n) \ge \sum_{i=1}^n a_i x_i^2 + \sum_{i=1}^n b_i x_i$ holds.

(B3) $\psi \in C^1(\mathbb{R}^n;\mathbb{R})$ and there exist $c_i \ge 0$, $1 \le i \le n$ such that $\psi(x_1, \cdots, x_n) \le \sum_{i=1}^n c_i |x_i|$ holds.

(B4) $f \in C^1(\mathbb{R}^n;\mathbb{R})$ and there exist $K_1 > 0$ and $\alpha_i \ge 0$, $1 \le i \le n$ such that $|f(x_1, \cdots, x_n)| \le K_1 e^{\sum_{i=1}^n \alpha_i x_i^2}$ holds. Moreover, we assume that $f(x^*) \neq 0$.

(B5) $k(\theta) \ge 0$ and $k(\theta) = o((\log(\theta))^2)$ hold.

Asymptotic behavior

Since $Hess\phi(x^*)$ is a positive definite matrix, we may use the orthogonal matrix Q and the diagonal matrix Λ satisfying $Hess\phi(x^*) = Q\Lambda Q^T$ and we denote

$$Q = \begin{bmatrix} q_{1,1} & \cdots & q_{1,n} \\ \vdots & \ddots & \vdots \\ q_{n,1} & \cdots & q_{n,n} \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \qquad (2)$$

by their components, where $\lambda_i > 0$, $1 \le i \le n$ denote the eigenvalues of $Hess\phi(x^*)$.

Asymptotic behavior

Theorem 4

Assume (B). Define $w = \int_{\mathcal{C}} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} dx$, *where* $\mathcal{C}$ *is given by*

$$\mathcal{C} = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \middle| \sum_{k=1}^{n} \frac{q_{j_i,k}}{\sqrt{\lambda_k}} x_k \leq 0 \, (1 \leq i \leq m) \right\},$$

for $1 \leq m \leq n$ *and* $\mathcal{C} = \mathbb{R}^n$ *for* $m = 0$. *Then, we have*

$$I(\theta) \sim w \frac{f(x^*)}{|Hess\phi(x^*)|^{\frac{1}{2}}} \frac{e^{-\theta^2\phi(x^*) + k(\theta)\psi(x^*) + \frac{k(\theta)^2}{2\theta^2}\sum_{i=1}^{n}\frac{1}{\lambda_i}(\sum_{j=1}^{n}\partial_i\psi(x^*)q_{j,i})^2}}{\theta^n}, \theta \to \infty.$$

Some comments

The proof is done by using Taylor's theorem in order to pick up the main part and vanish the ignorable part of the multiple integral with some delicate estimates. Since the proof is long and seems to be uninteresting, we skip the proof, here.

Remark 1

- *Laplace's method for multiple integrals are studied by [Hsu1](Duke Math. J. 1948), [Hsu2](Amer. J. Math. 1951) and [Hsu3](Quart. J. Oxford 1951) in various cases, however, the assumption in this talk have not been considered(as far as I know).*

Asymptotic behavior

Corollary 5

Assume (B1)-(B4) and $k(\theta)=\theta$. Moreover, we assume that ϕ and ψ are quadratic and linear functions, respectively. Then one has

$$I(\theta) \sim w \frac{f(x^*)}{|Hess\phi(x^*)|^{\frac{1}{2}}} \frac{e^{-\theta^2\phi(x^*)+\theta\psi(x^*)+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}(\sum_{j=1}^{n}\partial_i\psi(x^*)q_{j,i})^2}}{\theta^n}, \theta \to \infty$$

where w is defined as $w=\int_{\mathcal{C}} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} dx$ with $\mathcal{C}$ given by

$$\mathcal{C} = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \middle| \sum_{k=1}^{n} \frac{q_{j_i,k}}{\sqrt{\lambda_k}} \left(x_k + \sum_{j=1}^{n} \partial_j\psi(x^*) \frac{q_{j,k}}{\sqrt{\lambda_k}} \right) \leq 0 \; (1 \leq i \leq m) \right\},$$

if $1 \leq m \leq n$ and $\mathcal{C}=\mathbb{R}^n$ if $m=0$.

Asymptotic behavior

For $n \geq 2$, let $(X_1, \cdots, X_n)$ be a random vector with the distribution $N(\mu, V_n^{-1})$ defined on $(\Omega, \mathcal{F}, P)$. We assume that $(X_1, \cdots, X_n)$ is not degenerate, thus,

$$\begin{aligned} P\left(X_1 \in dx_1, \cdots, X_n \in dx_n\right) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{\sqrt{|V_n^{-1}|}} e^{-\frac{1}{2}(x-\mu)^T V_n (x-\mu)} dx_1 \cdots dx_n \\ &=: p(x_1, \cdots, x_n) dx_1 \cdots dx_n \end{aligned} \tag{3}$$

holds, where we have defined $V_n = [v_{i,j}^n]_{1 \leq i,j \leq n} = [Cov(X_i, X_j)]_{1 \leq i,j \leq n}^{-1}$ and $\mu = [\mu_1, \cdots, \mu_n]$, $\mu_i := E^P[X_i]$ for $1 \leq i \leq n$. Define $M_n = \max\{X_1, \cdots, X_n\}$, then since it holds that

$$P\left(M_n \leq \theta\right) = P\left(X_1 \leq \theta, \cdots, X_n \leq \theta\right) = \int_{-\infty}^{\theta} \cdots \int_{-\infty}^{\theta} p(x_1, \cdots, x_n) dx_1 \cdots dx_n,$$

the density function of M_n is given by ($d\hat{x}_k := dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n$)

$$\begin{aligned} p_{M_n}(\theta) &= \sum_{k=1}^{n} \int_{-\infty}^{\theta} \cdots \int_{-\infty}^{\theta} p(x_1, \cdots, x_{k-1}, \theta, x_{k+1}, \cdots, x_n) d\hat{x}_k \\ &=: \sum_{k=1}^{n} J_k(\theta). \end{aligned}$$

Asymptotic behavior

Then, the change of variables $\theta y_i = x_i, i \neq k$ yields

$$J_k(\theta) = \theta^{n-1} \int_{-\infty}^{1} \cdots \int_{-\infty}^{1} p(\theta x_1, \cdots, \theta x_n) d\hat{x}_k$$

$$= \theta^{n-1} \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{\sqrt{|V_n^{-1}|}} \int_{-\infty}^{1} \cdots \int_{-\infty}^{1} e^{-\frac{1}{2}\sum_{i,j=1}^{n} v_{i,j}^n (\theta x_i - \mu_i)(\theta x_j - \mu_j)} d\hat{x}_k$$

$$= \frac{\theta^{n-1}}{(2\pi)^{\frac{n}{2}} \sqrt{|V_n^{-1}|}} e^{-\frac{1}{2}\left[v_{k,k}^n (\theta - \mu_k)^2 - 2(\theta - \mu_k)\sum_{i \neq k}^{n} v_{i,k}^n \mu_i + \sum_{i,j \neq k}^{n} v_{i,j}^n \mu_i \mu_j\right]}$$

$$\times \int_{-\infty}^{1} \cdots \int_{-\infty}^{1} e^{-\theta^2\left[\sum_{i \neq k}^{n} v_{i,k}^n x_i + \frac{1}{2}\sum_{i,j \neq k}^{n} v_{i,j}^n x_i x_j\right] + \theta\left[\mu_k \sum_{i \neq k}^{n} v_{i,k}^n x_i + \sum_{i,j \neq k}^{n} v_{i,j}^n \mu_j x_i\right]} d\hat{x}_k$$

We define two functions on $\mathbb{R}^{n-1}$ by ($\hat{x}_k := (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)$)

$$\phi_k(\hat{x}_k) = \frac{1}{2}\sum_{i,j \neq k}^{n} v_{i,j}^n x_i x_j + \sum_{i \neq k}^{n} v_{i,k}^n x_i, \tag{4}$$

$$\psi_k(\hat{x}_k) = \mu_k \sum_{i \neq k}^{n} v_{i,k}^n x_i + \sum_{i,j \neq k}^{n} v_{i,j}^n \mu_j x_i = \sum_{i \neq k}^{n} \left(\sum_{j=1}^{n} v_{i,j}^n \mu_j\right) x_i.$$

Asymptotic behavior

For $n \times n$-matrix V_n, define $V_n^{(-k)}$ by $(n-1) \times (n-1)$-matrix given by removing the k-th row and k-th column from V_n.
Now, we note that $V_n^{(-k)} = Hess\phi_k(\hat{x}_k)$ and $V_n^{(-k)}$ may be represented as $V_n^{(-k)} = Q_k \Lambda_k Q_k^T$, where $Q_k := [q_{i,j}^k]_{1 \le i,j \neq k \le n}$ is an orthogonal matrix and Λ_k is a diagonal matrix with its diagonal components λ_i^k for $1 \le i \neq k \le n$.

Asymptotic behavior

Corollary 6

Let $\phi_k(\hat{x}_k)$ be defined by (4). Assume that for $1 \le k \le n$, $V_n^{(-k)}$ is a positive definite matrix and $\hat{x}_k \mapsto \phi_k(\hat{x}_k)$ has a unique global minimum at $\hat{x}_k^* \in (-\infty, 1] \times \cdots \times (-\infty, 1]$. Then, there exist constants $w_k > 0$, $1 \le k \le n$ such that

$$p_{M_T^n}(\theta) \sim \frac{|V_n|^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\sum_{i,j=1}^{n} v_{i,j}^n \mu_i \mu_j}$$

$$\times \sum_{k=1}^{n} \frac{w_k}{|V_n^{(-k)}|^{\frac{1}{2}}} e^{-\left[\frac{v_{k,k}^n}{2} + \phi_k(\hat{x}_k^*)\right]\theta^2 + \left[\sum_{i=1}^{n} v_{i,k}^n \mu_i + \psi_k(\hat{x}_k^*)\right]\theta + \frac{1}{2}\sum_{i \ne k}^{n} \frac{1}{\lambda_i^k}\left(\sum_{l=1}^{n} v_{i,l}^n \mu_l\right)^2 \left(\sum_{j \ne k}^{n} q_{j,i}^k\right)^2},$$

as $\theta \to \infty$ holds.

Asymptotic behavior

Proposition 1

For $n \geq 2$*, let* $\{a_i\}_{1 \leq i \leq n}$ *and* $\{b_i\}_{1 \leq i \leq n}$ *be two sequences in* $\mathbb{R}$*. Define* $g(i,j) = a_i b_j - a_j b_i$ *for* $1 \leq i \leq j \leq n$*. Assume that* $b_i \neq 0$ *for* $1 \leq i \leq n$ *and* $g(i,j) \neq 0$ *for* $1 \leq i < j \leq n$*. Then the determinant of the matrix,*

$$[a_{i \vee j} b_{i \wedge j}]_{1 \leq i,j \leq n} = \begin{bmatrix} a_1 b_1 & a_2 b_1 & \cdots & a_n b_1 \\ a_2 b_1 & a_2 b_2 & \cdots & a_n b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & a_n b_n \end{bmatrix} \tag{5}$$

is given by $a_n b_1 \prod_{i=2}^{n} g(i-1, i)$*.*

Asymptotic behavior

Proposition 1

Moreover, if $a_n \neq 0$ then the inverse matrix of (5) is given by the following tridiagonal matrix,

$$[a_{i\vee j}b_{i\wedge j}]^{-1}_{1\le i,j\le n} = \begin{bmatrix} \frac{b_2}{b_1 g(1,2)} & \frac{-1}{g(1,2)} & & & \\ \frac{-1}{g(1,2)} & \frac{g(1,3)}{g(1,2)g(2,3)} & \frac{-1}{g(2,3)} & & \\ & \frac{-1}{g(2,3)} & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & \frac{-1}{g(n-2,n-1)} & \frac{g(n-2,n)}{g(n-2,n-1)g(n-1,n)} & \frac{-1}{g(n-1,n)} \\ & & & \frac{-1}{g(n-1,n)} & \frac{a_{n-1}}{a_n g(n-1,n)} \end{bmatrix}. \tag{6}$$

Asymptotic behavior

Lemma 7

For $n \times n$-matrix defined by (5), suppose that $b_i \neq 0$ for $1 \leq i \leq n$, $g(i,j) \neq 0$ for $i < j$ and $a_n \neq 0$. Let $V_n = [v^n_{i,j}]_{1 \leq i,j \leq n}$ be defined by (6).(i.e. We define $[v^n_{i,j}]_{1 \leq i,j \leq n} = [a_{i \vee j} b_{i \wedge j}]^{-1}_{1 \leq i,j \leq n}$.)
Define ϕ_k on $\mathbb{R}^{n-1}$ by (4), that is

$$\phi_k(\hat{x}_k) = \frac{1}{2} \sum_{i,j \neq k}^{n} v^n_{i,j} x_i x_j + \sum_{i \neq k}^{n} v^n_{i,k} x_i$$

for $1 \leq k \leq n$. Then, ϕ_k attains its global minimum at

$$\hat{x}^*_k = \left(\frac{b_1}{b_k}, \cdots, \frac{b_{k-1}}{b_k}, \frac{a_{k+1}}{a_k}, \cdots, \frac{a_n}{a_k} \right)$$

and we have

$$\phi_k(\hat{x}^*_k) = \begin{cases} -\frac{a_2}{2a_1 g(1,2)}, & (k = 1), \\ -\frac{1}{2}\left(\frac{b_{k-1}}{b_k g(k-1,k)} + \frac{a_{k+1}}{a_k g(k,k+1)} \right), & (2 \leq k \leq n-1) \\ -\frac{b_{n-1}}{2b_n g(n-1,n)}, & (k = n). \end{cases}$$

Asymptotic behavior

Remark 2

We may regard (6) as $V_n = [Cov(X_i, X_j)]^{-1}_{1\le i,j\le n}$ *in (3).*

Example

- Itô process with deterministic integrands

$$X_t = x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s$$

- Brownian bridge

$$Z_t = \left\{ \begin{array}{ll} a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\int_0^t \frac{1}{T-s}dW_s, & 0 \le t < T, \\ b, & t = T \end{array} \right\}$$

- Ornstein-Uhlenbeck process

$$U_t = U_0 - \alpha \int_0^t U_s ds + \sigma W_t, t \in [0, \infty)$$

Asymptotic behavior

- Itô process with deterministic integrands
  $\rightarrow Cov(X_{t_i}, X_{t_j}) = \int_0^{t_i} \sigma^2(s)ds =: f(0,i), i \leq j$
- Brownian bridge$\rightarrow$ $Cov(Z_{t_i}, Z_{t_j}) = t_i(1 - t_j/T), i \leq j$
- Ornstein-Uhlenbeck process$\rightarrow$ $Cov(U_{t_i}, U_{t_j}) = \frac{\sigma^2}{2\alpha} e^{-\alpha t_j}(e^{\alpha t_i} - e^{-\alpha t_i}), i \leq j$

Correspondence table

	Itô with deterministic integrands	Brownian bridge	O-U process
a_i	1	1-t_i/T	$\frac{\sigma^2}{2\alpha} e^{-\alpha t_i}$
b_i	$f(0,i)$	t_i	$e^{\alpha t_i} - e^{-\alpha t_i}$

for $1 \leq i \leq n$.

Asymptotic behavior

About $\hat{x}_k^*$

- Itô process with deterministic integrands
  $$\rightarrow \hat{x}_k^* = \left(\frac{f(0,1)}{f(0,k)}, \cdots, \frac{f(0,k-1)}{f(0,k)}, 1 \cdots, 1\right) \in \underbrace{(-\infty, 1] \times \cdots \times (-\infty, 1]}_{(n-1)-\text{times}}$$
- Brownian bridge
  $$\rightarrow \hat{x}_k^* = \left(\frac{t_1}{t_k}, \cdots, \frac{t_{k-1}}{t_k}, \frac{T-t_{k+1}}{T-t_k}, \cdots, \frac{T-t_n}{T-t_k}\right) \in \underbrace{(-\infty, 1) \times \cdots \times (-\infty, 1)}_{(n-1)-\text{times}}$$
- Ornstein-Uhlenbeck process
  $$\rightarrow \hat{x}_k^* = \left(\frac{e^{\alpha t_1} - e^{-\alpha t_1}}{e^{\alpha t_k} - e^{-\alpha t_k}}, \cdots, \frac{e^{\alpha t_{k-1}} - e^{-\alpha t_{k-1}}}{e^{\alpha t_k} - e^{-\alpha t_k}}, e^{-\alpha(t_{k+1}-t_k)}, \cdots, e^{-\alpha(t_n - t_k)}\right) \in \underbrace{(-\infty, 1) \times \cdots \times (-\infty, 1)}_{(n-1)-\text{times}}$$

Asymptotic behavior (Itô process with deterministic integrands)

Let $\{X_t, t \in [0,\infty)\}$ satisfy

$$X_t = x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s, t \in [0,\infty),$$

where $b, \sigma : [0,\infty) \to \mathbb{R}$ are deterministic functions and consider $M_T^n = \max\{X_{t_1}, \cdots, X_{t_n}\}$.
We assume,

Assumption (C)

(C1) $\int_0^T |b(s)|ds < \infty$ for arbitrary $T > 0$.

(C2) For arbitrary $T > 0$, $\int_0^T |\sigma(s)|^2 ds < \infty$ holds and there exists $c > 0$ such that $|\sigma(t)| > c$ for any $t \in [0,\infty)$.

We take $a_i = 1$ and $b_i = f(0,i)$ for $1 \le i \le n$ in Proposition 1 and Lemma 7 and obtain $g(i,j) = f(i,j)$ for $1 \le i < j \le n$. Moreover, we have $\mu_i = X_0 + \int_0^{t_i} b(s)ds$ for $1 \le i \le n$ and we define $\mu_0 = 0$, for simplicity of

Asymptotic behavior (Itô process with deterministic integrands)

Theorem 8

Assume (C). Then the density function of M_T^n has the following asymptotic behavior,

$$p_{M_T^n}(\theta) \sim \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{\prod_{i=1}^n f(i-1,i)^{\frac{1}{2}}} e^{-\frac{1}{2}\left[\sum_{i=1}^{n-1} \mu_i \left(\frac{\mu_i-\mu_{i-1}}{f(i-1,i)} - \frac{\mu_{i+1}-\mu_i}{f(i,i+1)}\right) + \frac{\mu_n(\mu_n-\mu_{n-1})}{f(n-1,n)}\right]}$$

$$\times \left\{ \sum_{k=1}^{n-1} \frac{w_k e^{-\frac{\theta^2}{2f(0,k)} + \frac{\mu_k}{f(0,k)}\theta + \frac{1}{2}\left[\sum_{i\neq k}^{n-1} \frac{1}{\lambda_i^k}\left(\frac{\mu_i-\mu_{i-1}}{f(i-1,i)} - \frac{\mu_{i+1}-\mu_i}{f(i,i+1)}\right)^2 \left(\sum_{j\neq k}^n q_{j,i}^k\right)^2 + \frac{1}{\lambda_n^k}\left(\frac{\mu_n-\mu_{n-1}}{f(n-1,n)}\right)^2 \left(\sum_{j\neq k}^n q_{j,n}^k\right)^2\right]}}{|V_n^{(-k)}|^{\frac{1}{2}}} \right.$$

$$\left. + \frac{w_n}{|V_n^{(-n)}|^{\frac{1}{2}}} e^{-\frac{\theta^2}{2f(0,n)} + \frac{\mu_n}{f(0,n)}\theta + \frac{1}{2}\sum_{i=1}^{n-1} \frac{1}{\lambda_i^n}\left(\frac{\mu_i-\mu_{i-1}}{f(i-1,i)} - \frac{\mu_{i+1}-\mu_i}{f(i,i+1)}\right)^2 \left(\sum_{j=1}^{n-1} q_{j,i}^n\right)^2} \right\}, \tag{7}$$

where w_k, $1 \leq k \leq n-1$ are defined by

Asymptotic behavior (Itô process with deterministic integrands)

Theorem 8

$$w_k = \int_{\mathcal{C}_k} e^{-\frac{1}{2}\sum_{i\neq k}^{n} x_i^2} d\hat{x}_k,$$

$$\mathcal{C}_k = \left\{ \hat{x}_k \in \mathbb{R}^{n-1} \middle| \sum_{l\neq k}^{n} \frac{q_{j,l}^k}{\sqrt{\lambda_l^k}} \left[x_l + \frac{1}{\sqrt{\lambda_l^k}} \left(\sum_{m\neq k}^{n-1} \left(\frac{\mu_m - \mu_{m-1}}{f(m-1,m)} - \frac{\mu_{m+1} - \mu_m}{f(m,m+1)} \right) q_{m,l}^k \right.\right.\right.$$

$$\left.\left.\left. + \frac{\mu_n - \mu_{n-1}}{f(n-1,n)} q_{n,l}^k \right) \right] \leq 0, j = k+1, \cdots, n \right\}$$

and $w_n := \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2}\sum_{i=1}^{n-1} x_i^2} d\hat{x}_n = (2\pi)^{\frac{n-1}{2}}$.

Asymptotic behavior (Brownian bridge)

Let us consider the process satisfying

$$Z_t = \left\{ \begin{array}{ll} a\left(1-\frac{t}{T}\right) + b\frac{t}{T} + (T-t)\int_0^t \frac{1}{T-s}dW_s, & 0 \leq t < T, \\ b, & t = T \end{array} \right\}$$

where $a, b \in \mathbb{R},\ T > 0$ and define $\hat{M}_T^n = \{Z_{t_1}, \cdots, X_{t_n}\}$. We take $a_i = 1 - t_i/T$ and $b_i = t_i$ and we note $\mu_i = E^P[Z_{t_i}] = a + (b-a)t_i/T$.

Asymptotic behavior (Brownian bridge)

Theorem 9

The density function of $\hat{M}_T^n$, $p_{\hat{M}_T^n}(\theta)$ *satisfies*

$$p_{\hat{M}_T^n}(\theta)$$

$$\sim \frac{1}{(2\pi)^{\frac{1}{2}}\left(1-\frac{t_n}{T}\right)^{\frac{1}{2}}\prod_{i=1}^n (t_i - t_{i-1})^{\frac{1}{2}}} e^{-\frac{1}{2}\left[\sum_{i=1}^{n-1}\mu_i\left(\frac{\mu_i-\mu_{i-1}}{t_i-t_{i-1}}-\frac{\mu_{i+1}-\mu_i}{t_{i+1}-t_i}\right)+\mu_n\left(\frac{\mu_n-\mu_{n-1}}{t_n-t_{n-1}}+\frac{\mu_n}{t_{n+1}-t_n}\right)\right]}$$

$$\left\{\sum_{k=1}^{n-1}\frac{e^{-\frac{t_{n+1}\theta^2}{2t_k(t_{n+1}-t_k)}+\frac{\mu_k t_{n+1}\theta}{t_k(t_{n+1}-t_k)}+\frac{1}{2}\left[\sum_{i\neq k}^{n-1}\frac{1}{\lambda_i^k}\left(\frac{\mu_i-\mu_{i-1}}{t_i-t_{i-1}}-\frac{\mu_{i+1}-\mu_i}{t_{i+1}-t_i}\right)^2\left(\sum_{j\neq k}^n q_{j,i}^k\right)^2+\frac{1}{\lambda_n^k}\left(\frac{\mu_n-\mu_{n-1}}{t_n-t_{n-1}}+\frac{\mu_n}{t_{n+1}-t_n}\right)^2\left(\sum_{j\neq k}^n q_{j,n}^k\right)^2\right]}}{|V_n^{(-k)}|^{\frac{1}{2}}}\right.$$

$$\left.+\frac{1}{|V_n^{(-n)}|^{\frac{1}{2}}} e^{-\frac{t_{n+1}\theta^2}{2t_n(t_{n+1}-t_n)}+\frac{\mu_n t_{n+1}\theta}{t_n(t_{n+1}-t_n)}+\frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{\lambda_i^n}\left(\frac{\mu_i-\mu_{i-1}}{t_i-t_{i-1}}-\frac{\mu_{i+1}-\mu_i}{t_{i+1}-t_i}\right)^2\left(\sum_{j=1}^{n-1}q_{j,i}^n\right)^2}\right\}.$$

Asymptotic behavior (O-U process)

Let U satisfy

$$U_t = U_0 - \alpha \int_0^t U_s ds + \sigma W_t, t \in [0, \infty) \tag{8}$$

where $\alpha, \sigma > 0$ and define $\tilde{M}_T^n = \max\{U_{t_1}, \cdots, U_{t_n}\}$. We take $a_i = \frac{\sigma^2}{2\alpha} e^{-\alpha t_i}$ and $b_i = e^{\alpha t_i} - e^{-\alpha t_i}$ and we note $\mu_i = E^P[U_{t_i}] = U_0 e^{-\alpha t_i}$.

Asymptotic behavior (O-U process)

Theorem 10

The density function of $\tilde{M}_T^n$, $p_{\tilde{M}_T^n}(\theta)$ *satisfies*

$$p_{\tilde{M}_T^n}(\theta) \sim \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{(2\alpha)^{\frac{n}{2}} e^{\frac{\alpha}{2} t_n}}{\sigma^n \prod_{i=1}^n \left(e^{\alpha(t_i - t_{i-1})} - e^{-\alpha(t_i - t_{i-1})}\right)^{\frac{1}{2}}} \exp\left[-\frac{\alpha U_0^2}{\sigma^2(e^{2\alpha t_1} - 1)}\right]$$

$$\times \left\{ \frac{1}{|V_n^{(-1)}|^{\frac{1}{2}}} \exp\left[-\frac{\alpha}{\sigma^2(1 - e^{-2\alpha t_1})}\theta^2 + \frac{2\alpha U_0}{\sigma^2(e^{\alpha t_1} - e^{-\alpha t_1})}\theta\right]\right.$$

$$\left. + \sum_{k=2}^n \frac{1}{|V_n^{(-k)}|^{\frac{1}{2}}} \exp\left[-\frac{\alpha\theta^2}{\sigma^2(1 - e^{-\alpha t_k})} + \frac{2\alpha U_0 \theta}{\sigma^2(e^{\alpha t_k} - e^{-\alpha t_k})} + \frac{2\alpha^2 U_0^2}{\lambda_1^k \sigma^4 (e^{\alpha t_1} - e^{-\alpha t_1})^2} \left(\sum_{j \neq k}^n q_{j,1}^k\right)^2\right]\right\}.$$

Asymptotic behavior (Joint density function)

Consider the random vector $(\max\{X_1, \cdots, X_n\}, X_{n+1})$, where $(X_1, \cdots, X_{n+1}) \sim N(\mu, V_{n+1}^{-1})$. We define $\tilde{V}$ by $n \times n$-matrix given by removing the $(n+1)$-th row and the $(n+1)$-th column from $V_{n+1} = [Cov(X_i, X_j)]_{1 \leq i,j \leq n+1}^{-1}$. We note that $\tilde{V}^{(-k)}$ can be represented as $\tilde{V}^{(-k)} = \tilde{Q}_k \tilde{\Lambda}_k \tilde{Q}_k^T$, where $\tilde{Q}_k = [\tilde{q}_{i,j}^k]_{1 \leq i,j \neq k \leq n}$ is an orthogonal matrix and $\tilde{\Lambda}_k = diag\{\tilde{\lambda}_1^k, \cdots, \tilde{\lambda}_{k-1}^k, \tilde{\lambda}_{k-1}^k, \cdots, \tilde{\lambda}_n^k\}$.

Asymptotic behavior (Itô process with deterministic integrands)

Theorem 11

Assume (C) and let $\eta \in \mathbb{R}$ *be fixed. Then, the joint density function of* (M_T^n, X_T) *satisfies*

$$p_{M_T^n, X_T}(\theta,\eta) \sim \frac{1}{2\pi}\frac{1}{\prod_{i=1}^{n+1} f(i-1,i)^{\frac{1}{2}}} e^{-\frac{(\eta-\mu_{n+1})^2}{2f(n,n+1)} - \frac{\mu_n \eta}{f(n,n+1)} - \sum_{i=1}^n \left(\frac{\mu_i^2}{2f(i-1,i)} + \frac{\mu_i^2}{2f(i,i+1)}\right) + \sum_{i=1}^n \frac{\mu_i \mu_{i+1}}{f(i,i+1)}}$$

$$\Bigg[\sum_{k=1}^{n-1} \Bigg(\frac{e^{\frac{1}{2}\sum_{i\neq k}^n \frac{1}{\tilde{\lambda}_i^k}\left(\frac{\mu_i-\mu_{i-1}}{f(i-1,i)} - \frac{\mu_{i+1}-\mu_i}{f(i,i+1)}\right)^2 \left(\sum_{j\neq k}^n \tilde{q}_{j,i}^k\right)^2 + \frac{\eta^2\left(\sum_{j\neq k}^n \tilde{q}_{j,n}^k\right)^2}{2f(n,n+1)^2\tilde{\lambda}_n^k} + \frac{\eta\left(\frac{\mu_n-\mu_{n-1}}{f(n-1,n)} - \frac{\mu_{n+1}-\mu_n}{f(n,n+1)}\right)\left(\sum_{j\neq k}^n \tilde{q}_{j,n}^k\right)^2}{f(n,n+1)\tilde{\lambda}_n^k}}}{|\tilde{V}^{(-k)}|^{\frac{1}{2}}}$$

$$\times\ e^{-\frac{\theta^2}{2}\left(\frac{1}{f(0,k)} + \frac{1}{f(k,n+1)}\right) + \theta\left[\frac{\eta}{f(k,n+1)} + \frac{\mu_k}{f(0,k)} - \frac{\mu_{n+1}-\mu_k}{f(k,n+1)}\right]}\Bigg)$$

$$+ \frac{e^{\frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{\tilde{\lambda}_i^n}\left(\frac{\mu_i-\mu_{i-1}}{f(i-1,i)} - \frac{\mu_{i+1}-\mu_i}{f(i,i+1)}\right)^2\left(\sum_{j=1}^{n-1}\tilde{q}_{j,i}^n\right)^2 - \frac{\theta^2}{2}\left(\frac{1}{f(0,n)} + \frac{1}{f(n,n+1)}\right) + \theta\left[\frac{\eta}{f(n,n+1)} + \frac{\mu_n}{f(0,n)} - \frac{\mu_{n+1}-\mu_n}{f(n,n+1)}\right]}}{|\tilde{V}^{(-n)}|^{\frac{1}{2}}}\Bigg].$$

Asymptotic behavior (O-U process)

Theorem 12

Let $\eta \in \mathbb{R}$ be fixed. Then the joint density function of $(\tilde{M}_T^n, U_T)$, $p_{\tilde{M}_T^n, U_T}$ satisfies

$$p_{\tilde{M}_T^n, U_T}(\theta, \eta) \sim \frac{(2\alpha)^{\frac{n+1}{2}} e^{\frac{\alpha t_{n+1}}{2}}}{2\pi\sigma^{n+1} \prod_{i=1}^{n+1} \left(e^{\alpha(t_i - t_{i-1})} - e^{-\alpha(t_i - t_{i-1})}\right)^{\frac{1}{2}}} e^{-\frac{a_n}{a_{n+1} g(n,n+1)}\eta^2 - \frac{2\alpha^2 U_0^2 a_1}{\sigma^4 b_1}}$$

$$\times \left[\frac{1}{|\tilde{V}^{(-1)}|} e^{-\frac{b_{n+1}}{2b_1 g(1,n+1)}\theta^2 + \frac{\eta\theta}{g(1,n+1)} + \frac{\eta^2}{2\tilde{\lambda}_n^1 g(n,n+1)^2}\left(\sum_{j=2}^n \tilde{q}_{j,n}^1\right)^2} \right.$$

$$+ \sum_{k=2}^{n-1} \frac{1}{|\tilde{V}^{(-k)}|} e^{-\frac{b_{n+1}}{2b_k g(k,n+1)}\theta^2 + \left(\frac{\eta}{g(k,n+1)} + \frac{2\alpha U_0}{\sigma^2 b_k}\right)\theta + \frac{2\alpha^2 U_0^2}{\tilde{\lambda}_1^k \sigma^4 b_1^2}\left(\sum_{j\neq k}^n \tilde{q}_{j,1}^k\right)^2 + \frac{\eta^2}{2\tilde{\lambda}_n^k g(n,n+1)^2}\left(\sum_{j\neq k}^n \tilde{q}_{j,n}^k\right)^2}$$

$$\left. + \frac{1}{|\tilde{V}^{(-n)}|} e^{-\frac{b_{n+1}}{2b_n g(n,n+1)}\theta^2 + \left(\frac{\eta}{g(n,n+1)} + \frac{2\alpha U_0}{\sigma^2 b_n}\right)\theta + \frac{2\alpha^2 U_0^2}{\tilde{\lambda}_1^n \sigma^4 b_1^2}\left(\sum_{j=1}^{n-1} \tilde{q}_{j,1}^n\right)^2} \right],$$

where $a_i = \frac{\sigma^2}{2\alpha} e^{-\alpha t_i}$ and $b_i = e^{\alpha t_i} - e^{-\alpha t_i}$ for $1 \leq i \leq n+1$.

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