On density function concerning discrete time maximum of some one-dimensional diffusion processes

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1 Introduction

In this talk, we will show some results on the density functions related to discrete time maximum of some one-dimensional diffusion processes. That is defined by $M_T^n = \max\{X_{t_1}, \cdots, X_{t_n}\}$ for a fixed time interval $[0, T]$ and a time partition $\Delta_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < t_{n+1} = T$ for $n \geq 2$, where $\{X_t, t \in [0, \infty)\}$ denotes a one-dimensional diffusion process.

Firstly, we shall deal with the following one-dimensional stochastic differential equation (SDE),

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, t \in [0, \infty) \tag{1}$$

where $x_0 \in \mathbb{R}$, $b, \sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions and $\{W_t, t \in [0, \infty)\}$ is a one-dimensional standard Brownian motion.

The first goal of this talk is to prove an integration by parts (IBP) formula for the random vector (M_T^n, X_T). That is the formula of the form $E[\partial_\beta \varphi(M_T^n, X_T)] = E[\varphi(M_T^n, X_T)H_\beta]$ for a smooth function φ, where H_β is a certain random variable and $E[\cdot]$ denotes the expectation with respect to a certain probability measure. Then, we will apply the IBP formula to study on the density function of (M_T^n, X_T).

The second goal is to obtain asymptotic behaviors of the density functions of M_T^n and (M_T^n, X_T) for Gaussian processes. For this purpose, we shall consider the following multiple integral,

$$I(\theta) := \int_R f(x_1, \cdots, x_n)e^{-\theta^2\phi(x_1,\cdots,x_n)+k(\theta)\psi(x_1,\cdots,x_n)}dx_1 \cdots dx_n, \tag{2}$$

where $R = \prod_{i=1}^n(-\infty, d_i]$, $d_i \in \mathbb{R}$ for $1 \leq i \leq n$ and $f, \phi, \psi : \mathbb{R}^n \to \mathbb{R}$ are measurable functions, then obtain the asymptotic behavior of $I(\theta)$ as $\theta \to \infty$ by using the Laplace's method. The result will be used to obtain the asymptotic behaviors of the density functions. The process satisfying (1) where b, σ do not depend on the space parameter, Brownian Bridge and Ornstein-Uhlenbeck process will be considered as the examples.

2 Main results

For b, σ of (1), we assume the following,

Assumption (A)

(A1) For $t \in [0, \infty)$, $b(t, \cdot), \sigma(t, \cdot) \in C_b^\infty(\mathbb{R}; \mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t, \cdot)$ and $\sigma(t, \cdot)$ do not depend on t. In particular, let $c(\sigma)$ be a constant which bounds $|\sigma(t, x)|$.

(A2) There exists $c > 0$ such that

$$|\sigma(t, x)| \geq c$$

holds, for any $x \in \mathbb{R}$ and $t \in [0, \infty)$.

Theorem 1. *Assume (A). Let $G \in \mathbb{D}^\infty$. Then, for any multi index $\beta \in \{1, 2\}^k$, $k \geq 1$, there exists $H_\beta(G) \in \mathbb{D}^\infty$ such that*

$$E^P[\partial_\beta \varphi(M_T^n, X_T)G] = E^P[\varphi(M_T^n, X_T)H_\beta(G)] \tag{3}$$

holds for arbitrary $\varphi \in C_b^\infty(\mathbb{R}^2; \mathbb{R})$.

For $f, \phi, \psi, k(\theta)$ of (2), we assume the following,
Assumption (B)

(B1) $\phi \in C^2(\mathbb{R}^n;\mathbb{R})$ and ϕ attains its global minimum at a point $x^* = (x_1^*, \cdots, x_n^*) \in R$, in particular, we assume that $x_{j_1}^* = d_{j_1}, \cdots, x_{j_m}^* = d_{j_m}$ for $1 \le j_1 < \cdots < j_m \le n$, $0 \le m \le n$ and $x_i^* < d_i$ for other $1 \le i \le n$.

(B2) There exist $a_i > 0$ and $b_i \in \mathbb{R}$, $1 \le i \le n$ such that $\phi(x_1, \cdots, x_n) \ge \sum_{i=1}^n a_i x_i^2 + \sum_{i=1}^n b_i x_i$ holds.

(B3) $\psi \in C^1(\mathbb{R}^n;\mathbb{R})$ and there exist $c_i \ge 0$, $1 \le i \le n$ such that $\psi(x_1, \cdots, x_n) \le \sum_{i=1}^n c_i |x_i|$ holds.

(B4) $f \in C^1(\mathbb{R}^n;\mathbb{R})$ and there exist $K_1 > 0$ and $\alpha_i \ge 0$, $1 \le i \le n$ such that $|f(x_1, \cdots, x_n)| \le K_1 e^{\sum_{i=1}^n \alpha_i x_i^2}$ holds. Moreover, we assume that $f(x^*) \neq 0$.

(B5) $k(\theta) \ge 0$ and $k(\theta) = o((\log(\theta))^2)$ hold.

Since $Hess\phi(x^*)$ is a positive definite matrix, we may use the orthogonal matrix Q and the diagonal matrix Λ satisfying $Hess\phi(x^*) = Q\Lambda Q^T$ and we denote these components

$$Q = \begin{bmatrix} q_{1,1} & \cdots & q_{1,n} \\ \vdots & \ddots & \vdots \\ q_{n,1} & \cdots & q_{n,n} \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \tag{4}$$

where $\lambda_i > 0$, $1 \le i \le n$ denote the eigenvalues of $Hess\phi(x^*)$.

The main theorem in this section is following.

Theorem 2. *Assume (B). Define* $w = \int_{\mathcal{C}} e^{-\frac{1}{2}\sum_{i=1}^n x_i^2} dx$, *where* $\mathcal{C}$ *is given by*

$$\mathcal{C} = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{k=1}^n \frac{q_{j_i,k}}{\sqrt{\lambda_k}} x_k \le 0 \; (1 \le i \le m) \right\}, \tag{5}$$

for $1 \le m \le n$ *and* $\mathcal{C} = \mathbb{R}^n$ *for* $m = 0$. *Then, we have*

$$I(\theta) \sim w \frac{f(x^*)}{|Hess\phi(x^*)|^{\frac{1}{2}}} \frac{e^{-\theta^2 \phi(x^*) + k(\theta)\psi(x^*) + \frac{k(\theta)^2}{2\theta^2} \sum_{i=1}^n \frac{1}{\lambda_i} (\sum_{j=1}^n \partial_i \psi(x^*) q_{j,i})^2}}{\theta^n}, \theta \to \infty. \tag{6}$$

References

[1] Fulks, W.: A generalization of Laplace's method. Proc. Amer. Math. Soc. **2**(4), 613-622 (1951).

[2] Hayashi, M., Kohatsu-Higa, A.: Smoothness of the distribution of the supremum of a multi-dimensional diffusion process. Potential Anal. **38** (1), 57-77 (2013).

[3] Hsu, L.C.: A theorem on the asymptotic behavior of a multiple integral. Duke Math. J. **15**(3), 623-632 (1948).

[4] Hsu, L.C.: On the asymptotic behavior of a class of multiple integrals involving a parameter. Amer. J. Math. **73**(3), 625-634 (1951).

[5] Hsu, L.C.: The asymptotic behavior of a kind of multiple integrals involving a parameter. Quart. J. Oxford **2**(1), 175-188 (1951).

[6] Nakatsu, T.: Integration by parts formulas concerning maxima of some SDEs with applications to study on density functions, preprint.

[7] Nakatsu, T.: On density function concerning discrete time maximum of some one-dimensional diffusion processes, preprint.

[8] Nualart, D.: The Malliavin Calculus and Related Topics, 2nd edn. Probability and its Applications (New York), Springer-Verlag, Berlin (2006).

[9] Shigekawa, I.: Stochastic analysis. Translations of Mathematical Monographs, vol. 224. American Mathematical Society (2004).