Error analysis for approximations to one-dimensional SDEs via perturbation method

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Consider 1-dim SDEs driven by fBm

Consider an SDE

$$X_t = x_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, d^\circ B_s,$$

where

- $x_0 \in \mathbf{R}$,
- $b, \sigma : \mathbf{R} \to \mathbf{R}$, smooth, bdd,
- B: 1-dim. fBm with the Hurst $1/3 < H < 1$
  which is defined on a prob. sp. $(\Omega, \mathcal{F}, \boldsymbol{P})$,
- $d^\circ B$: the symmetric integral
  (roughly speaking, the Stratonovich integral).

Use symbols $X_t^{x_0,B}$ and X_t^B to emphasize x_0 and B.

Consider three approximation schemes

Consider

- the Euler scheme $X^{\mathrm{Eul}(m)}$,
- the Milstein type scheme $X^{\mathrm{Mil}(m)}$,
- the Crank-Nicholson scheme $X^{\mathrm{CN}(m)}$

for the dyadic partition $\tau_k^m = k2^{-m}$ $(0 \leqslant k \leqslant 2^m)$.

Remark

- All schemes define stochastic processes from $[0, 1]$ to $\mathbf{R}$.

We omit the superscript m and denote the above by X^{Eul}, X^{Mil}, X^{CN} and τ_k.

Show a limit thm for an approximation error

For $X^{\text{App}(m)} = X^{\text{Eul}(m)}, X^{\text{Mil}(m)}, X^{\text{CN}(m)}$, we will show

$$(2^m)^{\text{rate}}\{X^{\text{App}(m)} - X\} \xrightarrow[m\to\infty]{} \text{proc.}$$

for some

- positive rate
- non-trivial proc.
- topology

Preceding results

- Neuenkirch-Nourdin('07)
    - the Euler scheme,
    - $1/2 < H < 1$.
- Gradinaru-Nourdin('09)
    - the Milstein type scheme,
    - $0 < H < 1$, $b \equiv 0$.
- N('15+)
    - the Crank-Nicholson scheme,
    - $1/3 < H < 1/2$, $b \equiv 0$.

The Crank-Nicholson scheme

Assume that m is large s.t. $\dfrac{1}{2^m} \leqslant \dfrac{1}{2\sup|b'|}$.
For m, define a subset $\Omega^{\mathrm{CN}} \equiv \Omega^{\mathrm{CN}(m)} \subset \Omega$ by

$$\Omega^{\mathrm{CN}} = \left\{ \sup_{|s-t|\leqslant 1/2^m} |B_t - B_s| \leqslant \frac{1}{2\sup|\sigma'|} \right\}.$$

Then

$$\lim_{m\to\infty} \boldsymbol{P}(\Omega^{\mathrm{CN}(m)}) = 1.$$

The Crank-Nicholson scheme

On Ω^{CN}, define $X^{\mathrm{CN}} : [0, 1] \to \mathbf{R}$ by the solution to

$$\begin{cases} X_0^{\mathrm{CN}} = x_0, \\ X_t^{\mathrm{CN}} = X_{\tau_{k-1}}^{\mathrm{CN}} + \frac{1}{2}\left\{b(X_t^{\mathrm{CN}}) + b(X_{\tau_{k-1}}^{\mathrm{CN}})\right\}(t - \tau_{k-1}) \\ \qquad\qquad + \frac{1}{2}\left\{\sigma(X_t^{\mathrm{CN}}) + \sigma(X_{\tau_{k-1}}^{\mathrm{CN}})\right\}(B_t - B_{\tau_{k-1}}) \\ \qquad\qquad\qquad\qquad \text{for } \tau_{k-1} < t \leqslant \tau_k. \end{cases}$$

Otherwise, set

$$X_t^{\mathrm{CN}} = x_0 \quad \text{for } 0 \leqslant t \leqslant 1.$$

Notation

Set

- $w = \sigma b' - \sigma' b$,
- $J_t = \exp\left(\int_0^t b'(X_s)\,ds + \int_0^t \sigma'(X_s)\,d^\circ B_s\right)$.

Assumption

$$\inf \sigma > 0.$$

CLT for the error of the CN scheme

Theorem (Aida and N.)

Let $1/3 < H < 1/2$. *We have*

$$\lim_{m\to\infty} 2^{m(3H-1/2)}\{X^{\mathrm{CN}(m)} - X\} = \sigma(X)U + J\int_0^{\bullet} J_s^{-1} U_s \mathrm{w}(X_s)\, ds$$

weakly wrt the uniform norm.

Theorem (Conti.)

Here, we define U by

$$U_t = \sigma_{3,H} \int_0^t f_3(X_s)\, dW_s,$$

where

- $f_3 = (\sigma^2)''/24$,
- $\sigma_{3,H} > 0$,
- *W is a standard Bm independent of B.*

Remarks on the main theorem

Let $\nabla_h X_t^B$ be the directional derivative in h, i.e.

$$\nabla_h X_t^B = \frac{dX_t^{x_0, B+ah}}{da}\Big|_{a=0}.$$

By using the Jacobi proc. J, we have

$$\nabla_h X = \sigma(X)h + J\int_0^{\bullet} J_s^{-1} h_s \mathrm{w}(X_s)\, ds.$$

Hence we have a formal expression

$$\text{(the limit proc.)} = \text{“}\nabla_U X_t\text{”}$$

Idea of perturbation method

Find a stochastic process $\tilde{h} : [0, 1] \to \mathbf{R}$ s.t.

- piecewise linear
- solution $X^{x_0, B+\tilde{h}}$ to a perturbed SDE

$$X_t^{x_0,B+\tilde{h}} = x_0 + \int_0^t b(X_s^{x_0,B+\tilde{h}})\, ds + \int_0^t \sigma(X_s^{x_0,B+\tilde{h}})\, d^\circ (B + \tilde{h})_s$$

satisfies

$$X_{\tau_k}^{\mathrm{CN}} = X_{\tau_k}^{x_0,B+\tilde{h}}$$

for every $k = 0, 1, \ldots, 2^m$.

Set $\lambda_t = t$ for $0 \leqslant t \leqslant 1$.

Proposition

If $\inf \sigma > 0$, *then* $a \mapsto X_t^{x_0, B + a2^m\lambda}$ *is bijective for* $\forall x_0$.

Proof. Note

$$\frac{dX_t^{x_0, B+a2^m\lambda}}{da} = \nabla_{2^m\lambda} X_t^{x_0, B+a2^m\lambda}$$

$$= \sigma(X_t^{x_0, B+a2^m\lambda}) \int_0^t \exp\left(\int_s^t \left[\frac{\mathrm{w}}{\sigma} \right] (X_u^{x_0, B+a2^m\lambda})\, du \right) 2^m\, ds$$

$$> C2^m t,$$

where $C > 0$ is a const.

Proposition

If $\inf \sigma > 0$, *then* $\exists \tilde{h} \equiv \tilde{h}^{(m)} \equiv \tilde{h}^{(m)}(B)$ *s.t.*

- *piecewise linear,*
- $X^{\mathrm{CN}}_{\tau_k} = X^{x_0, B+\tilde{h}}_{\tau_k} \quad \forall k.$

For $0 \leqslant t < 1$, denote the shift operator by θ_t, i.e.

$$(\theta_t B)_u = B_{u+t} - B_t$$

for $0 \leqslant u \leqslant 1 - t$.

Proof. Definition of $\tilde{h}$.

- From the bijectivity, we see $\exists \tilde{\kappa}_1, \ldots, \tilde{\kappa}_{2^m}$ s.t.

$$X_{1/2^m}^{X_{\tau_{k-1}}^{\mathrm{CN}}, \theta_{\tau_{k-1}} B + \tilde{\kappa}_k 2^m \lambda} = X_{\tau_k}^{\mathrm{CN}}.$$

- Define $\tilde{h} : [0, 1] \to \mathbf{R}$ by

$$\tilde{h}(t) = \tilde{h}(\tau_{k-1}) + \tilde{\kappa}_k 2^m (t - \tau_{k-1})$$

for $\tau_{k-1} \leqslant t \leqslant \tau_k$.

Remark

- $\tilde{\kappa}_k$ is a functional of $(X_{\tau_{k-1}}^{\mathrm{CN}}, \theta_{\tau_{k-1}} B)$.
- $\tilde{h}$ is a functional of $\{(X_{\tau_{k-1}}^{\mathrm{CN}}, \theta_{\tau_{k-1}} B)\}_{k=1}^{2^m}$.

Check $X^{\mathrm{CN}}_{\tau_k} = X^{x_0, B+\tilde{h}}_{\tau_k}$.

- Note

$$\theta_{\tau_{k-1}}(B + \tilde{h})_u = (\theta_{\tau_{k-1}} B)_u + \tilde{\kappa}_k 2^m \lambda_u$$

  for $0 \leqslant u \leqslant 1/2^m$.

- From "the Markov property", the equality above and the definition of $\tilde{\kappa}_k$, we see

$$X^{x_0, B+\tilde{h}}_{\tau_k} = X^{X^{x_0,B+\tilde{h}}_{\tau_{k-1}}, \theta_{\tau_{k-1}}(B+\tilde{h})}_{1/2^m}$$
$$= X^{X^{\mathrm{CN}}_{\tau_{k-1}}, \theta_{\tau_{k-1}} B + \tilde{\kappa}_k 2^m \lambda}_{1/2^m} = X^{\mathrm{CN}}_{\tau_k}.$$

Convergence of the main term of the error

From the perturbation method, we see

$$2^{m(3H-1/2)}\{X_1^{\mathrm{CN}(m)} - X_1\} = 2^{m(3H-1/2)}\{X_1^{x_0, B+\tilde{h}^{(m)}} - X_1^{x_0, B}\}.$$

In order to show convergence of RHS, we consider

- expression of $\tilde{h}^{(m)}$,
- convergence of $2^{m(3H-1/2)}\tilde{h}^{(m)}$.

Proposition

Set

- $f_3 = (\sigma^2)''/24$, $f_4 = f_3'\sigma/2$,
- $\tilde{\kappa}_{\alpha,k} = f_\alpha(X^{CN}_{\tau_{k-1}})(B_{\tau_k} - B_{\tau_{k-1}})^\alpha$,
- $\tilde{h}^{(m)}_\alpha(t) = \tilde{h}^{(m)}_\alpha(\tau_{k-1}) + \tilde{\kappa}_{\alpha,k} \cdot 2^m(t - \tau_{k-1})$

  for $\tau_{k-1} \leqslant t \leqslant \tau_k$ *and* $\alpha = 3, 4$.

Then

$$\tilde{h}^{(m)} = \tilde{h}^{(m)}_3 + \tilde{h}^{(m)}_4 + \tilde{h}^{(m)}_{rem}.$$

Here $r^{(m)} = \|\tilde{h}^{(m)}_{rem}\|_\infty$ *satisfies*

$$\lim_{m\to\infty} 2^{m(3H-1/2)} r^{(m)} = 0.$$

Proof. From a long calculation, we see the identity.

Set

- $\kappa_{\alpha,k} = f_\alpha(X^B_{\tau_{k-1}})(B_{\tau_k} - B_{\tau_{k-1}})^\alpha$,
- $h^{(m)}_\alpha(t) = h^{(m)}_\alpha(\tau_{k-1}) + \kappa_{\alpha,k} \cdot 2^m(t - \tau_{k-1})$ for $\tau_{k-1} \leqslant t \leqslant \tau_k$ and $\alpha = 3, 4$,
- $h^{(m)} = h^{(m)}_3 + h^{(m)}_4$.

Proposition

We have

$$2^{m(3H-1/2)} h^{(m)} \to U = \sigma_{3,H} \int_0^\cdot f_3(X_s)\, dW_s$$

weakly wrt the uniform norm.

Proof. We use the fourth moment theorem.

Plan to show convergence of the error

In order to prove

$$2^{m(3H-1/2)}\{X_1^{B+\tilde{h}^{(m)}} - X_1^B\} \to \text{proc},$$

we decompose $X_1^{B+\tilde{h}^{(m)}} - X_1^B$ as

$$X_1^{B+\tilde{h}^{(m)}} - X_1^B = \nabla_{h^{(m)}} X_1^B + \{X_1^{B+\tilde{h}^{(m)}} - X_1^{B+h^{(m)}}\}$$
$$+ \{X_1^{B+h^{(m)}} - X_1^B - \nabla_{h^{(m)}} X_1^B\}.$$

From the decomposition, we see the following:

- The first term is the main term. In fact,

$$2^{m(3H-1/2)}\nabla_{h^{(m)}}X_1^B = \nabla_{2^{m(3H-1/2)}h^{(m)}}X_1^B \to \nabla_U X_1^B.$$

- The third term is negligible. In fact,

$$2^{m(3H-1/2)}|\text{the third term}| \leqslant C2^{m(3H-1/2)}\|h^{(m)}\|_\infty^2 \to 0.$$

- Show convergence of the second term.

Convergence of the second term

Proposition

$$\lim_{m\to\infty} 2^{m(3H-1/2)}\{X_1^{B+\tilde{h}^{(m)}} - X_1^{B+h^{(m)}}\} = 0.$$

We prove this proposition from the estimates

$$\delta^{(m)} = \max_{1\leqslant k\leqslant 2^m} |X_{\tau_k}^{\mathrm{CN}(m)} - X_{\tau_k}| \quad \text{and} \quad \|\tilde{h}^{(m)} - h^{(m)}\|_\infty.$$

Remark

The Lipschitz conti. of the sol. map $B \mapsto X_t^B$, i.e.

$$|X_t^{B+h} - X_t^B| \leqslant C\|h\|_\infty.$$

Take small $0 < \epsilon < H$.

Proposition D1

There exists a random variable $C_B > 0$ s.t.

$$\delta^{(m)} \leqslant C_B 2^{-m\{3(H-\epsilon)-1\}}.$$

Proof. The estimate follows from

- the definition of the CN scheme,
- the Hölder continuity of fBm.

Proposition H1

$$\|\tilde{h}^{(m)} - h^{(m)}\|_\infty \leqslant C_B \delta^{(m)} 2^{-m\{3(H-\epsilon)-1\}} + r^{(m)}.$$

Recall the definition of $\tilde{h}^{(m)}_\alpha$:

- $\tilde{\kappa}_{\alpha,k} = f_\alpha(X^{\text{CN}}_{\tau_{k-1}})(B_{\tau_k} - B_{\tau_{k-1}})^\alpha,$
- $\tilde{h}^{(m)}_\alpha(t) = \tilde{h}^{(m)}_\alpha(\tau_{k-1}) + \tilde{\kappa}_{\alpha,k} \cdot 2^m(t - \tau_{k-1})$
  for $\tau_{k-1} \leqslant t \leqslant \tau_k$ and $\alpha = 3, 4.$

and the definition of $h^{(m)}_\alpha$:

- $\kappa_{\alpha,k} = f_\alpha(X^{B}_{\tau_{k-1}})(B_{\tau_k} - B_{\tau_{k-1}})^\alpha,$
- $h^{(m)}_\alpha(t) = h^{(m)}_\alpha(\tau_{k-1}) + \kappa_{\alpha,k} \cdot 2^m(t - \tau_{k-1})$
  for $\tau_{k-1} \leqslant t \leqslant \tau_k$ and $\alpha = 3, 4,$

Proof.

- Note

$$
\begin{aligned}
&\|\tilde{h}_\alpha^{(m)} - h_\alpha^{(m)}\|_\infty \\
&\leqslant \sum_{k=1}^{2^m} |f_\alpha(X_{\tau_{k-1}}^{\mathrm{CN}(m)}) - f_\alpha(X_{\tau_{k-1}})||B_{\tau_k} - B_{\tau_{k-1}}|^\alpha \\
&\leqslant \sum_{k=1}^{2^m} C\delta^{(m)} \cdot \{C_B 2^{-m(H-\epsilon)}\}^\alpha \\
&= C_B \delta^{(m)} 2^{-m\{\alpha(H-\epsilon)-1\}}.
\end{aligned}
$$

- From the definitions $\tilde{h}^{(m)}$ and $h^{(m)}$, we see the estimate.

Proposition D2

$$\delta^{(m)} \leqslant C_B \delta^{(m)} 2^{-m\{3(H-\epsilon)-1\}} + Cr^{(m)} + C\|h^{(m)}\|_\infty.$$

Proof. The Hölder inequality yields

$$\begin{aligned}\delta^{(m)} &= \max_{1 \leqslant k \leqslant 2^m} |X^{B+h^{(m)}}_{\tau_k} - X^B_{\tau_k}| \\ &\leqslant \|X^{B+\tilde{h}^{(m)}} - X^{B+h^{(m)}}\|_\infty + \|X^{B+h^{(m)}} - X^B\|_\infty.\end{aligned}$$

The Lipschitz conti. of the sol. map and Prop. H1 yields

$$\begin{aligned}\delta^{(m)} &\leqslant C\|\tilde{h}^{(m)} - h^{(m)}\|_\infty + C\|h^{(m)}\|_\infty \\ &\leqslant C_B \delta^{(m)} 2^{-m\{3(H-\epsilon)-1\}} + Cr^{(m)} + C\|h^{(m)}\|_\infty.\end{aligned}$$

Proposition D3

For every $L \in \mathbf{N} \cup \{0\}$, there exists a r.v. $C_{B,L} > 0$ s.t.

$$\delta^{(m)} \leqslant C_{B,L} 2^{-m\{3(H-\epsilon)-1\}(L+1)} + C_{B,L} r^{(m)} + C_{B,L} \|h^{(m)}\|_\infty$$

Proof. By using Prop. D2 recursively and Prop. D1 (rough estimate of $\delta^{(m)}$), we see

$$\begin{aligned}\delta^{(m)} &\leqslant C_{B,L} \delta^{(m)} 2^{-m\{3(H-\epsilon)-1\}L} + C_{B,L} r^{(m)} + C_{B,L} \|h^{(m)}\|_\infty \\ &\leqslant C_{B,L} 2^{-m\{3(H-\epsilon)-1\}(L+1)} + C_{B,L} r^{(m)} + C_{B,L} \|h^{(m)}\|_\infty.\end{aligned}$$

Proposition H2

For every $L \in \mathbf{N} \cup \{0\}$, we have

$$\|\tilde{h}^{(m)} - h^{(m)}\|_\infty \leqslant C_{B,L} 2^{-m\{3(H-\epsilon)-1\}(L+2)} + C_{B,L} r^{(m)} + C_{B,L} 2^{-m\{3(H-\epsilon)-1\}} \|h^{(m)}\|_\infty.$$

Proof. Combining Prop. H1 and D3, we obtain

$$\begin{aligned}\|\tilde{h}^{(m)} - h^{(m)}\|_\infty &\leqslant C_B \delta^{(m)} 2^{-m\{3(H-\epsilon)-1\}} + r^{(m)} \\ &\leqslant C_{B,L} 2^{-m\{3(H-\epsilon)-1\}(L+2)} + C_{B,L} r^{(m)} \\ &\quad + C_{B,L} 2^{-m\{3(H-\epsilon)-1\}} \|h^{(m)}\|_\infty.\end{aligned}$$

Proposition

The second term with $2^{m(3H-1/2)}$ converges to 0, i.e.

$$\lim_{m\to\infty} 2^{m(3H-1/2)}\{X_1^{B+\tilde{h}^{(m)}} - X_1^{B+h^{(m)}}\} = 0.$$

Proof.

- The Lipschitz conti. of the sol. map and Prop. H2 yield

$$\begin{aligned}\|X^{B+\tilde{h}^{(m)}} - X^{B+h^{(m)}}\|_\infty &\leqslant C\|\tilde{h}^{(m)} - h^{(m)}\|_\infty \\ &\leqslant C_{B,L}2^{-m\{3(H-\epsilon)-1\}(L+2)} + C_{B,L}r^{(m)} \\ &\quad + C_{B,L}2^{-m\{3(H-\epsilon)-1\}}\|h^{(m)}\|_\infty.\end{aligned}$$

- Take large $L \in \mathbf{N} \cup \{0\}$ s.t.

$$\lim_{m\to\infty} 2^{m(3H-1/2)} 2^{-m\{3(H-\epsilon)-1\}(L+2)} = 0.$$

- We see

$$\lim_{m\to\infty} 2^{m(3H-1/2)} r^{(m)} = 0$$

and

$$\lim_{m\to\infty} 2^{m(3H-1/2)} 2^{-m\{3(H-\epsilon)-1\}} \|h^{(m)}\|_\infty = 0.$$

- Combining them we obtain the conclusion.

Thank you for your attention