Error analysis for approximations to one-dimensional SDEs via perturbation method *

Nobuaki Naganuma (Mathematical Institute, Tohoku University)

1 Introduction and main result

For a one-dimensional fractional Brownian motion (fBm) B with the Hurst $1/3 < H < 1$, we consider a one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,d^\circ B_s, \quad t \in [0,1], \tag{1}$$

where $x_0 \in \mathbf{R}$ is a deterministic initial value and $d^\circ B$ stands for the symmetric integral in the sense of Russo-Vallois. In order to approximate the solution to (1), we consider the Crank-Nicholson scheme as real-valued stochastic process on the interval $[0,1]$. In this talk, we study asymptotic error distributions of the scheme.

In what follows, we assume that the coefficients b and σ in (1) are smooth and they are bounded together with all their derivatives. We give the definition of the Crank-Nicholson scheme for the m-th dyadic partition $\{\tau_k^m = k2^{-m}\}_{k=0}^{2^m}$:

Definition 1.1 (The Crank-Nicholson scheme)**.** For every $m \in \mathbf{N}$, the Crank-Nicholson scheme $X^{\mathrm{CN}(m)} : [0,1] \to \mathbf{R}$ is defined by a solution to an equation

$$\begin{cases} X_0^{\mathrm{CN}(m)} = x_0, \\ X_t^{\mathrm{CN}(m)} = X_{\tau_{k-1}^m}^{\mathrm{CN}(m)} + \dfrac{1}{2}\left\{ b(X_{\tau_{k-1}^m}^{\mathrm{CN}(m)}) + b(X_t^{\mathrm{CN}(m)}) \right\}(t - \tau_{k-1}^m) \\ \qquad\qquad + \dfrac{1}{2}\left\{ \sigma(X_{\tau_{k-1}^m}^{\mathrm{CN}(m)}) + \sigma(X_t^{\mathrm{CN}(m)}) \right\}(B_t - B_{\tau_{k-1}^m}) \quad \text{for } \tau_{k-1}^m < t \le \tau_k^m. \end{cases}$$

Since the Crank-Nicholson scheme is an implicit scheme, we need to restrict the domain of it and assure the existence of a solution to the equation above. Roughly speaking, the existence of the solution is ensured for large m.

In order to state our main result concisely, we set $\mathsf{w} = \sigma b' - \sigma' b$ and

$$J_t = \exp\left(\int_0^t b'(X_u)\,du + \int_0^t \sigma'(X_u)\,d^\circ B_u \right).$$

We assume the following hypothesis in order to obtain an expression of the error of the scheme:

*This talk is based on a joint work with Professor Shigeki Aida.

Hypothesis 1.2. $\inf \sigma > 0$.

The following is our main result:

Theorem 1.3. *Assume that Hypothesis* 1.2 *is satisfied. For* $1/3 < H < 1/2$*, we have*

$$\lim_{m\to\infty} 2^{m(3H-1/2)}\{X^{\mathrm{CN}(m)} - X\} = \sigma(X)U + J\int_0^{\cdot} J_s^{-1}\mathsf{w}(X_s)U_s\,ds$$

weakly with respect to the uniform norm. Here U *a stochastic process defined by*

$$U_t = \sigma_{3,H}\int_0^t f_{0,3}(X_u)\,dW_u, \tag{2}$$

where $\sigma_{3,H}$ *is a positive constant,* $f_{0,3} = (\sigma^2)''/24$ *and* W *is a standard Brownian motion independent of* B*.*

2 Sketch of proof

We explain the concept of perturbation method and give a sketch of proof of our main theorem.

The idea of perturbation method is to find a piecewise linear stochastic process $\tilde{h} \equiv \tilde{h}^{(m)} : [0,1] \to \mathbf{R}$ such that $X^{x_0,B+\tilde{h}}_{\tau^m_k} = X^{\mathrm{CN}(m)}_{\tau^m_k}$ for every $k = 1,\ldots,2^m$, where $X^{x_0,B+\tilde{h}}$ is a solution to an SDE with the same initial value x_0 and a perturbed driver $B+\tilde{h}$, that is,

$$X^{x_0,B+\tilde{h}}_t = x_0 + \int_0^t b(X^{x_0,B+\tilde{h}}_s)\,ds + \int_0^t \sigma(X^{x_0,B+\tilde{h}}_s)\,d^\circ(B+\tilde{h})_s.$$

Under Hypothesis 1.2, we see unique existence of $\tilde{h}$ and obtain an expression of it.

From the expression of $\tilde{h}^{(m)}$ and the Lipschitz continuity of the solution map $B \mapsto X^{x_0,B}$, we construct a piecewise linear function $h \equiv h^{(m)} : [0,1] \to \mathbf{R}$ such that (a) $2^{m(3H-1/2)}h^{(m)}$ converges to U defined by (2) and (b) $\tilde{h}^{(m)} - h^{(m)}$ is negligible. We can show Assertion (a) by using the fourth moment theorem. Assertion (b) is a nontrivial part in our proof. In order to justify Assertion (b), we need the following step:

(D1) estimate $\delta^{(m)} = \max_{1\le k\le 2^m} |X^{\mathrm{CN}(m)}_{\tau^m_k} - X^{x_0,B}_{\tau^m_k}|$ from the definition of the scheme,

(H1) estimate $\|\tilde{h}^{(m)} - h^{(m)}\|_\infty$ by a quantity involving $\delta^{(m)}$ from the construction of $\tilde{h}^{(m)}$ and $h^{(m)}$,

(D2) estimate $\delta^{(m)}$ by a quantity involving $\delta^{(m)}$ itself from (H1),

(D3) show a sharp estimate of $\delta^{(m)}$ by using (D2) repeatedly and (D1),

(H2) show Assertion (b) from (D3) and (H1).

For simplicity, we explain how to see the asymptotic error distribution of $X^{\mathrm{CN}(m)}_1 - X^{x_0,B}_1$. By using the properties of $h^{(m)}$ and the decomposition

$$\begin{aligned} X^{\mathrm{CN}(m)}_1 - X^{x_0,B}_1 &= X^{x_0,B+\tilde{h}^{(m)}}_1 - X^{x_0,B}_1 \\ &= \nabla_{h^{(m)}}X^{x_0,B}_1 + \{X^{x_0,B+\tilde{h}^{(m)}}_1 - X^{x_0,B+h^{(m)}}_1\} + \{X^{x_0,B+h^{(m)}}_1 - X^{x_0,B}_1 - \nabla_{h^{(m)}}X^{x_0,B}_1\}, \end{aligned}$$

we see Theorem 1.3. In fact, Assertion (a) implies that the first term converges to a nontrivial process, that is, $2^{m(3H-1/2)}\nabla_{h^{(m)}}X^{x_0,B}_1 = \nabla_{2^{m(3H-1/2)}h^{(m)}}X^{x_0,B}_1 \to \nabla_U X^{x_0,B}_1$ as $m \to \infty$. The convergences of the second and third term to 0 follow from Assertion (a) and (b), respectively.