Stochastic Analysis and Related Topics 2015

The rates of the L^p-convergence of the Euler-Maruyama and the Wong-Zakai approximations of path-dependent stochastic differential equations under the Lipschitz condition

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1. Introduction

In this talk we consider the stochastic differential equation

$$\begin{cases} dX_t &= \sigma(t, X)dB_t + b(t, X)dt \\ X_0 &= \xi \end{cases}$$

where $\sigma : [0, T] \times C_b([0, T]; \mathbb{R}^d) \to \mathbb{R}^d \bigotimes \mathbb{R}^r$

$b : [0, T] \times C_b([0, T]; \mathbb{R}^d) \to \mathbb{R}^d$, $\quad B$: r-dim. B.m.

We assume the Borel measurability of σ and b.

Assume that for $t \in [0, T]$, $w, w' \in C([0, T]; \mathbb{R}^d)$

$$\begin{aligned} &|\sigma(t, w) - \sigma(t, w')|_{\mathbb{R}^d \bigotimes \mathbb{R}^r} + |b(t, w) - b(t, w')|_{\mathbb{R}^d} \\ &\leq K_T \|w - w'\|_{C([0,t];\mathbb{R}^d)}. \end{aligned}$$

(This implies $\sigma(t, w)$ depends only on $\{w_s : s \in [0, t]\}$.)

Then, the equation has the pathwise-unique solution.

Path-dependent stochastic differential equations appear not only in mathematical finance,
but also in a transformation of Markov-type stochastic differential equations with reflection.

Let

D: a connected domain in $\mathbb{R}^d$.

$\mathrm{Var}_{[0,t]}(w)$: the total variation of w on $[0,t]$

$B(x,r) := \{y \in \mathbb{R}^d; |x-y| < r\}$

Definition

For given $w \in C([0,T];\mathbb{R}^d)$ with $w_0 \in \overline{D}$,

$(\xi,\phi) \in C([0,T];\overline{D}) \times C([0,T];\mathbb{R}^d)$ is called

a solution of the Skorohod equation on D,

if $\phi_0 = 0$, ϕ has the bounded variation on $[0,T]$,

$$\xi_t = w_t + \phi_t, \qquad \mathrm{Var}_{[0,t]}(\phi) = \int_0^t \mathbb{I}_{\partial D}(\xi_s) d\mathrm{Var}_{[0,s]}(\phi),$$

and there exists $\mathbf{n} \in C([0,T];\mathbb{R}^d)$ s.t.

$$\mathbf{n}_t \in \bigcup_{r>0} \{\tilde{\mathbf{n}} \in \mathbb{R}^d; |\tilde{\mathbf{n}}| = 1, B(\xi_t - r\tilde{\mathbf{n}}, r) \cap D = \emptyset\},$$

$$t \in \{s \in [0,T]; \xi_s \in \partial D\}$$

$$\phi_t = \int_0^t \mathbf{n}_s d\mathrm{Var}_{[0,s]}(\phi).$$

We remark that

$\mathbf{n}_t$ is an inward unit normal vector of ∂D at ξ_t.

The reflected process of w is defined by ξ.

When the Skorohod equation has
  the existence and the uniqueness of the solution,
we call $\Gamma : w \mapsto \xi$ the Skorohod map.

If $\overline{D} = \{x = (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d; x^1 \geq 0\}$,
the Skorohod map is given by

$$(\Gamma w)_t = \left(w_t^1 - \inf_{s \in [0,t]} (w_s^1 \wedge 0), w_t^2, \ldots, w_t^d \right).$$

Hence, $\Gamma : C([0,T]; \mathbb{R}^d) \to C([0,T]; \overline{D})$ is
Lipschitz continuous.

Skorohod equations on general domains was studied by Tanaka (1979), Lions and Sznitman (1984) and Saisho (1987).

Generally, the Skorohod map is

not Lipschitz continuous.

However, even in the cases of such a general domain the Skorohod map is (1/2)-Hölder continuous.

Consider a stochastic differential equation with the reflecting boundary condition,

$$\begin{cases} dX_t &= \sigma(t, X_t)dB_t + b(t, X_t)dt + d\Phi_t \\ X_0 &= x \in \overline{D} \end{cases}$$

where Φ plays the role of the reflection of X on ∂D, i.e. $\Gamma(X - \Phi) = X$.

On the other hand, consider a path-dependent stochastic differential equation

$$\begin{cases} dY_t &= \sigma(t, (\Gamma Y)_t)dB_t + b(t, (\Gamma Y)_t)dt \\ Y_0 &= x \in \overline{D}. \end{cases}$$

There is a one-to-one correspondence

between X and Y.

Indeed, if Y is a solution to the path-dependent SDE, $X := \Gamma Y$ satisfies the SDE with reflection.

While, if X is a solution to the SDE with reflection, then Y defined by

$$Y_t = x + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds$$

satisfies the path-dependent SDE.

The one-to-one correspondence is originally introduced in the case of half spaces

by Anderson and Orey (1976).

In view of the one-to-one correspondence, we consider the path-dependent stochastic differential equations

$$\begin{cases} dX_t & = \sigma(t, X) dB_t + b(t, X) dt \\ X_0 & = \xi \end{cases}$$

and study the Euler-Maruyama and the Wong-Zakai approximations.

We remark that Słomiński have studied the rate in the case of SDEs with reflection on general domains.

2. Euler-Maruyama approximation

Let $\triangle := \{0 = t_0 < t_1 < \cdots < t_N = T\}$.

Define the approximations $\sigma_\triangle, b_\triangle$ of σ, b by

$$\sigma_\triangle(t,w) := \sigma(t_k,w),\ b_\triangle(t,w) := b(t_k,w), \quad t \in [t_k, t_{k+1}).$$

Consider the following stochastic differential equation.

$$\begin{cases} dX_t^{\mathsf{EM}} &= \sigma_\triangle(t, X^{\mathsf{EM}})dB_t + b_\triangle(t, X^{\mathsf{EM}})dt \\ X_0^{\mathsf{EM}} &= \xi. \end{cases}$$

Then, $X_t^{\mathsf{EM}} = \xi + \sum_{l=0}^{k} \sigma(t_l, X^{\mathsf{EM}})(B_{t\wedge t_{l+1}} - B_{t_l})$

$$+ \sum_{l=0}^{k} b(t_l, X^{\mathsf{EM}})(t \wedge t_{l+1} - t_l).$$

Hence, X^{EM} is the Euler-Maruyama approx. w.r.t. $\triangle$.

For a Hilbert space H and a positive number K,

$F_K(H) := \{h : [0,T] \times C_b([0,T];\mathbb{R}^d) \to H$ s.t.

(F1) $|h(t,w)|_H \leq K$ for $t \in [0,T],\ w \in C([0,T];\mathbb{R}^d)$.

(F2) $|h(t,w) - h(s,w)|_H$

$$\leq K(\sqrt{t-s} + \|w(\cdot + s) - w(s)\|_{C([0,t-s];\mathbb{R}^d)})$$

for $0 \leq s < t \leq T$ and $w \in C([0,T];\mathbb{R}^d)$.

(F3) $|h(t,w) - h(t,w')|_H \leq K\|w - w'\|_{C([0,t];\mathbb{R}^d)}$

for $t \in [0,T],\ w, w' \in C([0,T];\mathbb{R}^d).\}$

Then, we have the following theorem.

Theorem

Let $\sigma \in F_K(\mathbb{R}^d \otimes \mathbb{R}^r)$ and $b \in F_K(\mathbb{R}^d)$.

Then, for $p \in [1, \infty)$ there exists a constant C independent of $\triangle$ and N, such that

$$E\left[\left\|X - X^{\mathsf{EM}}\right\|^p_{C([0,T];\mathbb{R}^d)}\right]^{1/p} \leq C|\triangle|^{1/2}.$$

Our result coincides with Słomiński's result in the case of SDEs with reflection whose Skorohod map is Lipschitz continuous.

Remark:

The condition (F2) on the coefficients

(or something like that)

is necessary for the Euler-Maruyama approximation.

Indeed, consider the (trivial) 1-dim. SDE

$$\begin{cases} dX_t &= \mathbb{I}_{\mathbb{Q}}(t)dB_t \\ X_0 &= 0, \end{cases}$$

and let $\triangle = \left\{ \frac{kT}{2^N}; k \in \mathbb{N} \right\}$.

Then, the coefficient is Lipschitz continuous (in w)

and the solution $X_t = 0$ for $t \in [0, T]$.

However, $X_t^{\mathrm{EM}} = B_t$.

Proof. It is sufficient to prove the case that $p \geq 2$.

$$
\begin{aligned}
&E\left[\left\|X - X^{\mathsf{EM}}\right\|_{C([0,t];\mathbb{R}^d)}^p\right] \\
&\leq CE\left[\sup_{s\in[0,t]}\left|\int_0^s \left(\sigma(u,X) - \sigma_\triangle(u,X^{\mathsf{EM}})\right) dB_u\right|_{\mathbb{R}^d}^p\right] \\
&\quad + CE\left[\sup_{s\in[0,t]}\left|\int_0^s \left(b(u,X) - b_\triangle(u,X^{\mathsf{EM}})\right) du\right|_{\mathbb{R}^d}^p\right] \\
&\leq CE\left[\left(\int_0^t \left|\sigma(u,X) - \sigma_\triangle(u,X^{\mathsf{EM}})\right|_{\mathbb{R}^d\otimes\mathbb{R}^r}^2 du\right)^{p/2}\right] \\
&\quad + CE\left[\left(\int_0^t \left|b(u,X) - b_\triangle(u,X^{\mathsf{EM}})\right|_{\mathbb{R}^d} du\right)^p\right] \\
&\leq C\int_0^t E\left[\left|\sigma(u,X) - \sigma_\triangle(u,X^{\mathsf{EM}})\right|_{\mathbb{R}^d\otimes\mathbb{R}^r}^p\right] du \\
&\quad + C\int_0^t E\left[\left|b(u,X) - b_\triangle(u,X^{\mathsf{EM}})\right|_{\mathbb{R}^d}^p\right] du.
\end{aligned}
$$

When $u \in [t_k, t_{k+1})$,

$$
\begin{aligned}
&E\left[\left|\sigma(u,X) - \sigma_\triangle(u,X^{\mathsf{EM}})\right|^p_{\mathbb{R}^d\otimes\mathbb{R}^r}\right] \\
&= E\left[\left|\sigma(u,X) - \sigma(t_k,X^{\mathsf{EM}})\right|^p_{\mathbb{R}^d\otimes\mathbb{R}^r}\right] \\
&\le CE\left[\left|\sigma(u,X) - \sigma(u,X^{\mathsf{EM}})\right|^p_{\mathbb{R}^d\otimes\mathbb{R}^r}\right] \\
&\qquad\qquad + CE\left[\left|\sigma(u,X^{\mathsf{EM}}) - \sigma(t_k,X^{\mathsf{EM}})\right|^p_{\mathbb{R}^d\otimes\mathbb{R}^r}\right] \\
&\le CE\left[\left\|X - X^{\mathsf{EM}}\right\|^p_{C([0,u];\mathbb{R}^d)}\right] + C(u-t_k)^{p/2} \\
&\quad + CE\left[\|X^{\mathsf{EM}}(\cdot + t_k) - X^{\mathsf{EM}}(t_k)\|^p_{C([0,u-t_k];\mathbb{R}^d)}\right].
\end{aligned}
$$

On the other hand, for $u \in [t_k, t_{k+1})$

$$E\left[\|X^{\mathsf{EM}}(\cdot + t_k) - X^{\mathsf{EM}}(t_k)\|^p_{C([0,u-t_k];\mathbb{R}^d)}\right]$$

$$= E\left[\sup_{s\in[t_k,u]}\left|\sigma(t_k, X^{\mathsf{EM}})(B_s - B_{t_k}) + b(t_k, X^{\mathsf{EM}})(s - t_k)\right|^p_{\mathbb{R}^d}\right]$$

$$\leq C\left(E\left[\sup_{s\in[t_k,u]}|B_s - B_{t_k}|^p_{\mathbb{R}^d}\right] + (u - t_k)^p\right)$$

$$\leq C(u - t_k)^{p/2}.$$

Thus, we have

$$\int_0^t E\left[\left|\sigma(u,X)-\sigma_\triangle(u,X^{\mathsf{EM}})\right|^p_{\mathbb{R}^d\otimes\mathbb{R}^r}\right]du$$
$$\leq C\int_0^t E\left[\left\|X-X^{\mathsf{EM}}\right\|^p_{C([0,u];\mathbb{R}^d)}\right]^{1/p}du+C|\triangle|^{p/2}.$$

Similarly we have

$$\int_0^t E\left[\left|b(u,X)-b_\triangle(u,X^{\mathsf{EM}})\right|^p_{\mathbb{R}^d}\right]du$$
$$\leq C\int_0^t E\left[\left\|X-X^{\mathsf{EM}}\right\|^p_{C([0,u];\mathbb{R}^d)}\right]^{1/p}du+C|\triangle|^{p/2}.$$

By applying Gronwall's inequality, we obtain the assertion.

3. Wong-Zakai approximation

Let $A : C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$ s.t.

(A1) $\|A(w) - A(w')\|_{C([0,t];\mathbb{R}^d)} \leq K_A \|w - w'\|_{C([0,t];\mathbb{R}^d)}$

for $t \in [0,T]$, $w, w' \in C([0,T];\mathbb{R}^d)$.

(A2) $|A(w)_t - A(w)_s|_{\mathbb{R}^d}$

$$\leq K_A \Big(\sqrt{t-s} + \|w(\cdot + s) - w(s)\|_{C([0,t-s];\mathbb{R}^d)} \Big)$$

for $0 \leq s < t \leq T$, and $w \in C([0,T];\mathbb{R}^d)$.

(A3) $\mathrm{Var}_{[0,t]}(A(w)) \leq K_A (1 + \|w - w(0)\|_{C([0,t];\mathbb{R}^d)})$

for $t \in [0,T]$, $w \in C([0,T];\mathbb{R}^d)$.

Let $f \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ with bounded derivatives. Define $\Gamma : C([0,T]; \mathbb{R}^d) \to C([0,T]; \mathbb{R}^d)$ by

$$(\Gamma w)_t := f(t, w_t) + A(w)_t, \quad t \in [0,T],\ w \in C([0,T]; \mathbb{R}^d).$$

Then, we have the Lipschitz continuity of Γ.

By the assumptions

$$\begin{aligned} &|(\Gamma w)_t - (\Gamma w)_s|_{\mathbb{R}^d} \\ &\leq C\left(\sqrt{t-s} + \|w(\cdot + s) - w(s)\|_{C([0,t-s];\mathbb{R}^d)}\right) \end{aligned}$$

for $0 \leq s < t \leq T$, and $w \in C([0,T]; \mathbb{R}^d)$.

Let $\sigma \in C_b([0,T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d \bigotimes \mathbb{R}^r)$ s.t.

$\sigma(t,x,y)$ is differentiable w.r.t x and y,

and σ and the derivatives are Lipschitz continuous.

Let $b \in C_b([0,T] \times C([0,T]; \mathbb{R}^d); \mathbb{R}^d)$ s.t.

$$|b(t,w) - b(t,w')|_{\mathbb{R}^d} \le K_b \|w - w'\|_{C([0,t];\mathbb{R}^d)}$$

for $t \in [0,T]$, and $w, w' \in C([0,T]; \mathbb{R}^d)$.

Consider the SDE of the Stratonovich type

$$\begin{cases} dX_t &= \sigma(t, X_t, (\Gamma X)_t) \circ dB_t + b(t,X)dt \\ X_0 &= \xi. \end{cases}$$

Remark:

Generally, even if $\sigma(t, X)$ is predictable, the stochastic integral of Stratonovich type

$$\int_0^t \sigma(s, X) \circ dB_s = \left(\sum_{j=1}^r \int_0^t \sigma_{ij}(s, X) dB_s^j + \frac{1}{2} \sum_{j=1}^r \langle \sigma_{ij}(\cdot, X), B^j \rangle_t \right)_i$$

(where $\langle \cdot, \cdot \rangle$ is the quadratic variation) is not defined. The reason is that $\sigma(t, X)$ needs to be a semimartingale in order to define $\langle \sigma_{ij}(\cdot, X), B^j \rangle$.
On the other hand, we do not know whether $\sigma(t, X)$ is a semimartingale or not.

For given $\triangle := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ define the piecewise linear approximation $B^{\triangle}$ of B by

$$B_t^{\triangle} := B_{t_k} + \frac{t - t_k}{t_{k+1} - t_k}(B_{t_k+1} - B_{t_k}), \quad t \in [t_k, t_{k+1}).$$

We define the equation of the Wong-Zakai approximation by

$$\begin{cases} dX_t^{\mathsf{WZ}} &= \sigma(t, X_t^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_t)dB_t^{\triangle} + b(t, X^{\mathsf{WZ}})dt \\ X_0^{\mathsf{WZ}} &= \xi. \end{cases}$$

Then, we have the following theorem.

Theorem Let σ and b as above.

Then, for $p \in [1, \infty)$ there exists a constant C independent of $\triangle$ and N, such that

$$E\left[\left\|X - X^{\mathrm{WZ}}\right\|^p_{C([0,T];\mathbb{R}^d)}\right]^{1/p} \leq C|\triangle|^{1/2}(1 + \log N)^{1/2}.$$

Lemma

$$E\left[\sup_{s\in[t_k,t_{k+1}]}\left|X_s - X_{t_k}\right|_{\mathbb{R}^d}^q\right]^{1/q} \leq C_q|t_{k+1}-t_k|^{1/2}$$

$$E\left[\sup_{s\in[t_k,t_{k+1}]}\left|X_s^{\mathrm{WZ}} - X_{t_k}^{\mathrm{WZ}}\right|_{\mathbb{R}^d}^q\right]^{1/q} \leq C_q|t_{k+1}-t_k|^{1/2}$$

$\exists\varepsilon > 0$ s.t.

$$E\left[\exp\left(\varepsilon \sup_{s\in[t_k,t_{k+1}]}\frac{\left|X_s - X_{t_k}\right|_{\mathbb{R}^d}^2}{t_{k+1}-t_k}\right)\right] \leq C$$

$$E\left[\exp\left(\varepsilon \sup_{s\in[t_k,t_{k+1}]}\frac{\left|X_s^{\mathrm{WZ}} - X_{t_k}^{\mathrm{WZ}}\right|_{\mathbb{R}^d}^2}{t_{k+1}-t_k}\right)\right] \leq C.$$

From the lemma in the previous slide,

we have the following.

Lemma

$$E\left[\max_{k=0,1,\ldots,N-1}\sup_{s\in[t_k,t_{k+1}]}|X_s-X_{t_k}|^p_{\mathbb{R}^d}\right] \le C|\triangle|^{p/2}(1+\log N)^{p/2},$$

$$E\left[\max_{k=0,1,\ldots,N-1}\sup_{s\in[t_k,t_{k+1}]}|X^{\mathrm{WZ}}_s-X^{\mathrm{WZ}}_{t_k}|^p_{\mathbb{R}^d}\right] \le C|\triangle|^{p/2}(1+\log N)^{p/2}.$$

Let

$$U_t^i := \frac{1}{2} \sum_{j=1}^{r} \sum_{l=1}^{d} \int_0^t \frac{\partial \sigma_{ij}}{\partial x^l}(s, X_s, (\Gamma X)_s) \sigma_{lj}(s, X_s, (\Gamma X)_s) ds$$

$$V_t^i := \frac{1}{2} \sum_{j=1}^{r} \sum_{l=1}^{d} \sum_{m=1}^{d} \int_0^t \frac{\partial \sigma_{ij}}{\partial y^l}(s, X_s, (\Gamma X)_s) \frac{\partial f^l}{\partial x^m}(s, X_s) \times \sigma_{mj}(s, X_t, (\Gamma X)_t) ds.$$

Then, we have

$$\begin{aligned} & d(X_t - X_t^{\mathsf{WZ}}) \\ & = \sigma(t, X_t, (\Gamma X)_t) dB_t + \Big(b(t, X) - b(t, X^{\mathsf{WZ}})\Big) dt \\ & \quad + dU_t + dV_t - \sigma(t, X_t^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_t) dB_t^{\triangle}. \end{aligned}$$

By integration by parts formula,

$$\int_{t_k}^{t_{k+1}} \sigma_{ij}(s, X_s^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_s) dB_s^{\triangle,j}$$

$$= \int_{t_k}^{t_{k+1}} \sigma_{ij}(s, X_s^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_s) \frac{B_{t_{k+1}}^j - B_{t_k}^j}{t_{k+1} - t_k} ds$$

$$= \sigma_{ij}(t_k, X_{t_k}^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_{t_k})(B_{t_{k+1}}^j - B_{t_k}^j)$$

$$+ \int_{t_k}^{t_{k+1}} \frac{t_{k+1} - s}{t_{k+1} - t_k} \frac{\partial \sigma_{ij}}{\partial t}(s, X_s^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_s)(B_{t_{k+1}}^j - B_{t_k}^j) ds$$

$$+ \sum_{l=1}^{d} \int_{t_k}^{t_{k+1}} (t_{k+1} - s) \frac{\partial \sigma_{ij}}{\partial x^l}(s, X_s^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_s) \frac{B_{t_{k+1}}^j - B_{t_k}^j}{t_{k+1} - t_k} dX_s^{\mathsf{WZ},l}$$

$$+ \sum_{l=1}^{d} \int_{t_k}^{t_{k+1}} (t_{k+1} - s) \frac{\partial \sigma_{ij}}{\partial y^l}(s, X_s^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_s) \frac{B_{t_{k+1}}^j - B_{t_k}^j}{t_{k+1} - t_k} d(\Gamma X^{\mathsf{WZ}})_s^l.$$

Hence, we have

$$
\begin{aligned}
&X^i_{t_n} - X^{\mathsf{WZ},i}_{t_n} \\
&= \sum_{k=0}^{n-1} \sum_{j=1}^{r} \int_{t_k}^{t_{k+1}} \Big(\sigma_{ij}(s, X_s, (\Gamma X)_s) - \sigma_{ij}(t_k, X^{\mathsf{WZ}}_{t_k}, (\Gamma X^{\mathsf{WZ}})_{t_k}) \Big)\, dB^j_s \\
&+ \int_0^{t_n} \Big(b(s, X) - b(s, X^{\mathsf{WZ}}) \Big)\, ds \\
&- \sum_{k=0}^{n-1} \sum_{j=1}^{r} \int_{t_k}^{t_{k+1}} \frac{t_{k+1} - s}{t_{k+1} - t_k} \frac{\partial \sigma_{ij}}{\partial t}(s, X^{\mathsf{WZ}}_s, (\Gamma X^{\mathsf{WZ}})_s)(B^j_{t_{k+1}} - B^j_{t_k}) ds \\
&+ U^i_{t_n} - \sum_{k=0}^{n-1} \sum_{j=1}^{r} \sum_{l=1}^{d} \int_{t_k}^{t_{k+1}} (t_{k+1} - s) \frac{\partial \sigma_{ij}}{\partial x^l}(s, X^{\mathsf{WZ}}_s, (\Gamma X^{\mathsf{WZ}})_s) \frac{B^j_{t_{k+1}} - B^j_{t_k}}{t_{k+1} - t_k} dX^{\mathsf{WZ},l}_s \\
&+ V^i_{t_n} - \sum_{k=0}^{n-1} \sum_{j=1}^{r} \sum_{l=1}^{d} \int_{t_k}^{t_{k+1}} (t_{k+1} - s) \frac{\partial \sigma_{ij}}{\partial y^l}(s, X^{\mathsf{WZ}}_s, (\Gamma X^{\mathsf{WZ}})_s) \frac{B^j_{t_{k+1}} - B^j_{t_k}}{t_{k+1} - t_k} d(\Gamma X^{\mathsf{WZ}})^l_s \\
&=: I^i_1(t_n) + I^i_2(t_n) + I^i_3(t_n) + I^i_4(t_n) + I^i_5(t_n).
\end{aligned}
$$

We consider the estimates of I_1, I_2, . . . , I_5.

Lemma

$$E\left[\max_{k=0,1,\dots,n}\left|I_1(t_k)^i\right|^p\right]$$
$$\le C\left(|\triangle|^{p/2}+\int_0^{t_n}E\left[\sup_{u\in[0,s]}\left|X_u-X_u^{\mathsf{WZ}}\right|_{\mathbb{R}^d}^p\right]ds\right),$$
$$E\left[\max_{k=0,1,\dots,n}\left|I_2(t_k)^i\right|^p\right]\le C\int_0^{t_n}E\left[\sup_{u\in[0,s]}\left|X_u-X_u^{\mathsf{WZ}}\right|_{\mathbb{R}^d}^p\right]ds,$$
$$E\left[\max_{k=0,1,\dots,n}\left|I_3^i(t_k)\right|^p\right]\le C|\triangle|^{p/2}.$$

Let

$$\mu_t^{ijm} := \sum_{k=0}^{N-1} \sum_{l=1}^{d} \left(\frac{\partial \sigma_{ij}}{\partial x^l} \sigma_{lm}\right) (t_k, X_{t_k}^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_{t_k})$$
$$\times \Big[\delta_{jm}(t \wedge t_{k+1} - t \wedge t_k)$$
$$-(B_{t\wedge t_{k+1}}^j - B_{t\wedge t_k}^j)(B_{t\wedge t_{k+1}}^m - B_{t\wedge t_k}^m)\Big].$$

Then, μ_t^{ijm} is a martingale, and we have the following.

Lemma

$$E\left[\max_{k=0,1,\ldots,n} \left|\sum_{j=1}^{r} \sum_{m=1}^{r} \mu_{t_k}^{ijm}\right|^p\right] \le C|\triangle|^{p/2}.$$

After a long calculation, we have

$$
\begin{aligned}
&\left|I_4^i(t_n)\right| \\
&\leq C\int_0^{t_n}\sup_{u\in[0,s]}|X_u - X_u^{\mathrm{WZ}}|_{\mathbb{R}^d}ds + \frac{1}{2}\max_{k=0,1,\ldots,n}\left|\sum_{j=1}^{r}\sum_{m=1}^{r}\mu_{t_k}^{ijm}\right| \\
&\quad + C\sum_{k=0}^{n-1}(t_{k+1}-t_k)\left(|\triangle|^{1/2} + \sup_{s\in[t_k,t_{k+1}]}|X_s - X_{t_k}|_{\mathbb{R}^d}\right) \\
&\quad + C\sum_{k=0}^{n-1}\sum_{j=1}^{r}\sum_{m=1}^{r}|B_{t_{k+1}}^j - B_{t_k}^j||B_{t_{k+1}}^m - B_{t_k}^m| \\
&\qquad\qquad \times\left(|\triangle|^{1/2} + \sup_{s\in[t_k,t_{k+1}]}|X_s^{\mathrm{WZ}} - X_{t_k}^{\mathrm{WZ}}|_{\mathbb{R}^d}\right) \\
&\quad + C\sum_{j=1}^{r}\sum_{k=0}^{n-1}|B_{t_{k+1}}^j - B_{t_k}^j|(t_{k+1}-t_k).
\end{aligned}
$$

Thus, we obtain the following.

Lemma

$$E\left[\max_{k=0,1,\dots,n}\left|I_4^i(t_k)\right|^p\right] \le C\left(|\triangle|^{p/2}+\int_0^{t_n}E\left[\sup_{u\in[0,s]}\left|X_u-X_u^{\mathrm{WZ}}\right|_{\mathbb{R}^d}^p\right]ds\right).$$

Let

$$\nu_t^{ijm} := \sum_{k=0}^{n-1} \sum_{l=1}^{d} \sum_{q=1}^{d} \left(\frac{\partial \sigma_{ij}}{\partial y^l} \sigma_{qm} \right) (t_k, X_{t_k}^{\mathsf{WZ}}, (\Gamma X^{\mathsf{WZ}})_{t_k}) \frac{\partial f^l}{\partial x^q} (t_k, X_{t_k}^{\mathsf{WZ}}) \times \Big(\delta_{jm} (t \wedge t_{k+1} - t \wedge t_k) - (B^j_{t \wedge t_{k+1}} - B^j_{t \wedge t_k})(B^m_{t \wedge t_{k+1}} - B^m_{t \wedge t_k}) \Big).$$

Then, ν_t^{ijm} is a martingale, and we have the following.

Lemma

$$E \left[\max_{k=0,1,\ldots,n} \left| \sum_{j=1}^{r} \sum_{m=1}^{r} \nu_{t_k}^{ijm} \right|^p \right] \le C |\triangle|^{p/2}.$$

Similary to I^4, after a long calculation, we have

$$
\begin{aligned}
&\left|I_5^i(t_n)\right| \\
&\leq C\int_0^{t_n} \sup_{u\in[0,s]} |X_u - X_u^{\mathsf{WZ}}|_{\mathbb{R}^d} ds + \frac{1}{2}\max_{k=0,1,\ldots,n}\left|\sum_{j=1}^r\sum_{m=1}^r \nu_{t_k}^{ijm}\right| \\
&\quad + C\sum_{k=0}^{n-1}(t_{k+1}-t_k)\left(|\triangle|^{1/2} + \sup_{s\in[t_k,t_{k+1}]}|X_s - X_{t_k}|_{\mathbb{R}^d}\right) \\
&\quad + C\sum_{k=0}^{n-1}\sum_{j=1}^r\sum_{m=1}^r |B_{t_{k+1}}^j - B_{t_k}^j||B_{t_{k+1}}^m - B_{t_k}^m| \\
&\qquad\qquad \times\left(|\triangle|^{1/2} + \sup_{s\in[t_k,t_{k+1}]}|X_s^{\mathsf{WZ}} - X_{t_k}^{\mathsf{WZ}}|_{\mathbb{R}^d}\right) \\
&\quad + C\sum_{j=1}^r\left(\max_{k=0,1,\ldots,n-1}|B_{t_{k+1}}^j - B_{t_k}^j|\right)\left(T + \mathrm{Var}_{[0,T]}(A(X^{\mathsf{WZ}}))\right).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&E\left[\max_{k=0,1,\dots,n}\left|I_5^i(t_k)\right|^p\right]^{1/p} \\
&\leq C|\triangle|^{1/2} + C\int_0^{t_n} E\left[\sup_{u\in[0,s]}|X_u - X_u^{\mathsf{WZ}}|_{\mathbb{R}^d}^p\right]^{1/p} ds \\
&\quad + C|\triangle|^{1/2}(1+\log N)^{1/2} \\
&\qquad \times \left(T + E\left[\left(\mathrm{Var}_{[0,T]}(A(X^{\mathsf{WZ}}))\right)^{2p}\right]^{1/(2p)}\right).
\end{aligned}
$$

So, to obtain the desired estimate, we need

$$
E\left[\left(\mathrm{Var}_{[0,T]}(A(X^{\mathsf{WZ}}))\right)^{2p}\right]^{1/(2p)} \leq C.
$$

By the assumption (A3) we have

Lemma

$$E\left[\max_{k=0,1,\dots,n}\left|I_5^i(t_k)\right|^p\right] \leq C\left(|\triangle|^{p/2}(1+\log N)^{p/2}+\int_0^{t_n}E\left[\sup_{u\in[0,s]}\left|X_u-X_u^{\mathrm{WZ}}\right|_{\mathbb{R}^d}^p\right]ds\right).$$

By the estimates we obtain

$$E\left[\max_{k=0,1,\ldots,n}|X_{t_k}-X^{\mathsf{WZ}}_{t_k}|^p_{\mathbb{R}^d}\right]$$
$$\leq C\sum_{i=1}^d\left(E\left[\max_{k=0,1,\ldots,n}|I^i_1(t_k)|^p\right]+\cdots+E\left[\max_{k=0,1,\ldots,n}|I^i_5(t_k)|^p\right]\right)$$
$$\leq C\left(|\triangle|^{p/2}(1+\log N)^{p/2}+\int_0^{t_n}E\left[\sup_{u\in[0,s]}\left|X_u-X^{\mathsf{WZ}}_u\right|^p_{\mathbb{R}^d}\right]ds\right)$$

Hence, for $t\in[t_n,t_{n+1}]$, we have

$$E\left[\sup_{s\in[0,t]}|X_s-X^{\mathsf{WZ}}_s|^p_{\mathbb{R}^d}\right]$$
$$\leq C\left(|\triangle|^{p/2}(1+\log N)^{p/2}+\int_0^{t}E\left[\sup_{u\in[0,s]}\left|X_u-X^{\mathsf{WZ}}_u\right|^p_{\mathbb{R}^d}\right]ds\right).$$

Therefore, Gronwall's inequality yields the result.

Thank you for your attention!