The rates of the L^p-convergence of the Euler-Maruyama and the Wong-Zakai approximations of path-dependent stochastic differential equations under the Lipschitz condition

Seiichiro Kusuoka
(Okayama University)

In this talk, we consider the Euler-Maruyama and the Wong-Zakai approximations of path-dependent stochastic differential equations. We remark that the theorems below are applicable to the Markov type stochastic differential equations with reflecting boundary condition on sufficiently good domains.

First we consider the Euler-Maruyama approximation of path-dependent stochastic differential equations. Let $T > 0$ and let ξ be an $\mathbb{R}^d$-valued random variable. Consider the following stochastic differential equation

$$\begin{cases} dX_t &= \sigma(t,X)dB_t + b(t,X)dt \\ X_0 &= \xi \end{cases} \tag{1}$$

where σ is an $\mathbb{R}^d \otimes \mathbb{R}^r$-valued function on $[0,T] \times C_b([0,T];\mathbb{R}^d)$, b is an $\mathbb{R}^d$-valued function on $[0,T] \times C_b([0,T];\mathbb{R}^d)$ and B is the r-dimensional Brownian motion. We assume the Lipschitz continuity of the coefficients in the following sense.

$$\begin{aligned} |\sigma(t,w) - \sigma(t,w')|_{\mathbb{R}^d \otimes \mathbb{R}^r} + |b(t,w) - b(t,w')|_{\mathbb{R}^d} \leq K_T \|w - w'\|_{C([0,t];\mathbb{R}^d)}, \\ t \in [0,T],\ w, w' \in C([0,T];\mathbb{R}^d) \end{aligned} \tag{2}$$

where K_T is a constant depending on T. Then, the solution X to (1) exists, and has the pathwise uniqueness. Let $\triangle := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ be a partition of the interval $[0,T]$. Define the approximations $\sigma_\triangle, b_\triangle$ of σ, b by

$$\sigma_\triangle(t,w) := \sigma(t_k, w),\ b_\triangle(t,w) := b(t_k,w), \quad t \in [t_k, t_{k+1})$$

for $k = 0, 1, \ldots, N-1$, and $w \in C([0,T];\mathbb{R}^d)$. We consider the following stochastic differential equation.

$$\begin{cases} dX_t^{\mathrm{EM}} &= \sigma_\triangle(t, X^{\mathrm{EM}})dB_t + b_\triangle(t, X^{\mathrm{EM}})dt \\ X_0^{\mathrm{EM}} &= \xi. \end{cases} \tag{3}$$

Then, (3) is the equation of the Euler-Maruyama approximation to (1). For a Hilbert space H and a positive number K, we define a class of H-valued functions $F_K(H)$ by the total set of $h : [0,T] \times C_b([0,T];\mathbb{R}^d) \to H$ such that

(F1) $|h(t,w)|_H \leq K$ for $t \in [0,T],\ w \in C([0,T];\mathbb{R}^d)$.

(F2) $|h(t,w) - h(s,w)|_H \leq K(\sqrt{t-s} + \|w(\cdot + s) - w(s)\|_{C([0,t-s];\mathbb{R}^d)})$
for $s, t \in [0,T]$ such that $s < t$, and $w \in C([0,T];\mathbb{R}^d)$.

(F3) $|h(t,w) - h(t,w')|_H \leq K\|w - w'\|_{C([0,t];\mathbb{R}^d)}$ for $t \in [0,T],\ w, w' \in C([0,T];\mathbb{R}^d)$.

Then, we have the following theorem.

Theorem 1. *Let $\sigma \in F_K(\mathbb{R}^d \otimes \mathbb{R}^r)$ and $b \in F_K(\mathbb{R}^d)$. Let X and X^{EM} be the solutions to (1) and to the equation of the Euler-Maruyama approximation (3), respectively. Then, for $p \in [1,\infty)$ there exists a constant C independent of $\triangle$ and N, such that*

$$E\left[\|X - X^{\mathrm{EM}}\|_{C([0,T];\mathbb{R}^d)}^p\right]^{1/p} \leq C|\triangle|^{1/2}.$$

Next we consider the Wong-Zakai approximation of path-dependent stochastic differential equations. Let $T>0$. Let A be a mapping from $C([0,T];\mathbb{R}^d)$ to $C([0,T];\mathbb{R}^d)$ such that

(A1) $\|A(w)-A(w')\|_{C([0,t];\mathbb{R}^d)} \le K_A\|w-w'\|_{C([0,t];\mathbb{R}^d)}$ for $t\in[0,T],\ w,w'\in C([0,T];\mathbb{R}^d)$.

(A2) $|A(w)_t-A(w)_s|_{\mathbb{R}^d} \le K_A\left(\sqrt{t-s}+\|w(\cdot+s)-w(s)\|_{C([0,t-s];\mathbb{R}^d)}\right)$
for $s,t\in[0,T]$ such that $s<t$, and $w\in C([0,T];\mathbb{R}^d)$.

(A3) $\mathrm{Var}_{[0,t]}(A(w)) \le K_A(1+\|w-w(0)\|_{C([0,t];\mathbb{R}^d)})$ for $t\in[0,T],\ w\in C([0,T];\mathbb{R}^d)$.

where $\mathrm{Var}_{[0,t]}(w)$ is the total variation of w on $[0,t]$, and let $f\in C^{1,2}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ which has the bounded derivatives. Define the mapping $\Gamma: C([0,T];\mathbb{R}^d)\to C([0,T];\mathbb{R}^d)$ by

$$(\Gamma w)_t := f(t,w_t)+A(w)_t, \quad t\in[0,T],\ w\in C([0,T];\mathbb{R}^d). \tag{4}$$

Then, we have the Lipschitz continuity of Γ in the sense of (2). From (A2) and (4), we have

$$|(\Gamma w)_t-(\Gamma w)_s|_{\mathbb{R}^d} \le (K_f+K_A)\left(\sqrt{t-s}+\|w(\cdot+s)-w(s)\|_{C([0,t-s];\mathbb{R}^d)}\right) \tag{5}$$

for $s,t\in[0,T]$ such that $s<t$, and $w\in C([0,T];\mathbb{R}^d)$, where K_f is a constant depending on the bounds of f and the derivatives of f. Let $\sigma\in C_b([0,T]\times\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R}^d\otimes\mathbb{R}^r)$ such that $\sigma(t,x,y)$ is differentiable with respect to x and y, and σ and the derivatives are Lipschitz continuous. Let $b\in C_b([0,T]\times C([0,T];\mathbb{R}^d);\mathbb{R}^d)$ such that there exists a positive constant K_b satisfying

$$|b(t,w)-b(t,w')|_{\mathbb{R}^d} \le K_b\|w-w'\|_{C([0,t];\mathbb{R}^d)}$$

for $t\in[0,T]$, and $w,w'\in C([0,T];\mathbb{R}^d)$. Let ξ be an $\mathbb{R}^d$-valued random variable. Consider the following stochastic differential equation of the Stratonovich type

$$\begin{cases} dX_t &= \sigma(t,X_t,(\Gamma X)_t)\circ dB_t + b(t,X)dt \\ X_0 &= \xi. \end{cases} \tag{6}$$

For a given partition $\triangle := \{0=t_0<t_1<\cdots<t_N=T\}$ of the interval $[0,T]$, we define the piecewise linear approximation $B^\triangle$ of B by

$$B_t^\triangle := B_{t_k} + \frac{t-t_k}{t_{k+1}-t_k}(B_{t_k+1}-B_{t_k}), \quad t\in[t_k,t_{k+1}).$$

We define the equation of the Wong-Zakai approximation to (6) by

$$\begin{cases} dX_t^{\mathrm{WZ}} &= \sigma(t,X_t^{\mathrm{WZ}},(\Gamma X^{\mathrm{WZ}})_t)dB_t^\triangle + b(t,X^{\mathrm{WZ}})dt \\ X_0^{\mathrm{WZ}} &= \xi. \end{cases} \tag{7}$$

Then, we have the following theorem.

Theorem 2. *Let σ and b as above. Let X and X^{WZ} be the solutions to (6) and to the equation of the Wong-Zakai approximation (7), respectively. Then, for $p\in[1,\infty)$ there exists a constant C independent of $\triangle$ and N, such that*

$$E\left[\|X-X^{\mathrm{WZ}}\|_{C([0,T];\mathbb{R}^d)}^p\right]^{1/p} \le C|\triangle|^{1/2}(1+\log N)^{1/2}.$$