Large deviations for rough path lifts of Donsker-Watanabe's δ-functions

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1 Aim

- $\exists$ a very general LDP of FW-type in Theorem 2.1, Takanobu-Watanabe '93.

- Prob. measures are not pushforwards of (scaled) Wiener measure, but of measures of finite energy.

- It can be regarded as a generalization of LDP of FW-type for scaled pinned diffusion measures.

$\not\exists$ a proof (It seems still open)

- We reformulate this LDP on
  geometric rough path space

- We give a rigorous proof by using RP theory, Malliavin calculus, quasi sure analysis.

- As a corollary, we obtain the LDP conjectured in TW '93, (thanks to Lyons' continuity thm & contraction principle).

[Remark] Elliptic case was done in I. '12+
We try (strongly) hypoelliptic case [much harder].

2 Background of Schilder/FW-type LDP on RP space

- Ledoux-Qian-Zhang '02. Schilder-type LDP for Brownian RP.
- In RP theory Itô map is conti. (Lyons' conti. thm.), from which FW-type LDP is immediate.
- Since then, LDP became a central topic in (the prob. aspect of) RP theory. $\exists$ Many papers.
- My previous work (I. '12+) was an attempt to extend LQZ's method to pinned diffusions. (didn't know TW). This work is a continuation.

3 Setting

$V_i : \mathbb{R}^n \to \mathbb{R}^n$ **vector fields** $(0 \leq i \leq d)$.

(A1): C^∞ **with bounded drivatives of all order** ≥ 1.

Consider the scaled SDE $(0 < \varepsilon \leq 1)$

$$dX_t^\varepsilon = \varepsilon \sum_{i=1}^{d} V_i(X_t^\varepsilon) \circ dw_t^i + \varepsilon^2 V_0(X_t^\varepsilon) dt$$

with $X_0^\varepsilon = x \in \mathbb{R}^n$.

Strong hypoellipticity condition everywhere:

$\Sigma_1 := \{V_i \mid 1 \le i \le d\}$ and
$\Sigma_k := \{[V_i, W] \mid 1 \le i \le d, W \in \Sigma_{k-1}\}$
for $k \ge 2$ recursively.
$\Sigma_k(x) := \{W(x) \mid W \in \Sigma_k\} \subset \mathbb{R}^n$ for $x \in \mathbb{R}^n$

(A2): For any $x \in \mathbb{R}^n$, $\cup_{k=1}^{\infty} \Sigma_k(x)$ spans $\mathbb{R}^n \cong T_x\mathbb{R}^n$ in the sense of linear algebra.

Note: V_0 is NOT involved.

♠ Under (A1)–(A2), X_t^ε is non-degenerate in the sense of Malliavin. $(\varepsilon > 0, t > 0)$

♠ Hence, $T(X_t^\varepsilon) = T \circ X_t^\varepsilon \in \tilde{\mathbb{D}}_{-\infty}$ is well-defined as a Watanabe distribution for $\forall T \in \mathcal{S}'(\mathbb{R}^n)$.

♠ In particular, $\exists$the heat kernel $p_t^\varepsilon(x, x') = \mathbb{E}[\delta_{x'}(X^\varepsilon(t, x))]$.

Note: $p_t^\varepsilon(x, x') > 0$ for $\varepsilon > 0, t > 0$, $x, x' \in \mathbb{R}^n$
($\because$ controllability of the skeleton ODE)

Skeleton ODE

$h \in \mathcal{H}$: a Cameron-Martin path.

$\phi_t = \phi(t, x, h)$ is a unique sol. of

$$d\phi_t = \sum_{i=1}^{d} V_i(\phi_t) dh_t^i, \qquad \phi_0 = x$$

No drift !

♠ Set $\mathcal{K}^{x,x'} := \{h \in \mathcal{H} \mid \phi(1, x, h) = x'\}$,
which is non-empty (controllability of the ODE)

Projection onto a linear subspace

$\mathcal{V}$: an l-dim. linear subspace of $\mathbb{R}^n$ $(1 \le l \le n)$

$\Pi_{\mathcal{V}} : \mathbb{R}^n \to \mathcal{V}$: the orthogonal projection.

$$Y_t^\varepsilon := \Pi_{\mathcal{V}}(X_t^\varepsilon), \qquad \psi(t,x,h) := \Pi_{\mathcal{V}}(\phi(t,x,h)),$$

$$\mathcal{M}^{x,a} := \{h \in \mathcal{H} \mid \psi(1,x,h) = a \in \mathcal{V}\}$$
$$= \bigcup \{\mathcal{K}^{x,x'} \mid x' \in \Pi_{\mathcal{V}}^{-1}(a)\} \neq \emptyset.$$

♠ For $a \in \mathcal{V}$, $\delta_a(Y_t^\varepsilon) = (\delta_a \circ \Pi_{\mathcal{V}})(X_t^\varepsilon)$ is well-defined as a positive Watanabe distribution. Hence, a finite measure on the Wiener space. ($\mathbb{E}[\delta_a(Y_t^\varepsilon)] > 0 \quad \Longrightarrow \exists$normalization)

Rough path space

$G\Omega^B_{\alpha,4m}(\mathbb{R}^d)$: geometric RP space with $(\alpha, 4m)$-Besov topology, where

$$m \in \mathbb{N}, \quad \frac{1}{3} < \alpha < \frac{1}{2} \quad s.t., \quad \alpha - \frac{1}{4m} > \frac{1}{3},$$
$$8m(\frac{1}{2} - \alpha) > 2.$$

♠ Besov-Hölder embedding.

$G\Omega^B_{\alpha,4m}(\mathbb{R}^d) \hookrightarrow G\Omega^H_{\alpha-(1/4m)}(\mathbb{R}^d)$

$$\|\mathbf{w}^1\|_{\alpha,4m-B} + \|\mathbf{w}^2\|_{2\alpha,2m-B}$$
$$:= \Big(\iint_{0\le s<t\le 1} \frac{|\mathbf{w}^1_{s,t}|^{4m}}{|t-s|^{1+4m\alpha}} dsdt\Big)^{1/4m}$$
$$+ \Big(\iint_{0\le s<t\le 1} \frac{|\mathbf{w}^2_{s,t}|^{2m}}{|t-s|^{1+4m\alpha}} dsdt\Big)^{1/2m}.$$

When $\mathrm{w} = (\mathrm{w}^1, \mathrm{w}^2)$ is Brownian RP,
(a power of) the above is a D_∞-functional.

$\Longrightarrow$ Cut-off within Watanabe's theory is available

Brownian rough path

$\mathcal{L} : C_0([0,1], \mathbb{R}^d) \to G\Omega^B_{\alpha,4m}(\mathbb{R}^d)$: the RP lift map via the dyadic polygonal approximations,

- $\mathcal{L}$ is defined outside a slim subset of Wiener sp.

$\Rightarrow (\varepsilon\mathcal{L})_*[\delta_a(Y_1^\varepsilon)]$ is a measure on $G\Omega^B_{\alpha,4m}(\mathbb{R}^d)$.

- $w \mapsto \mathcal{L}(w) =: \mathbf{W} = (\mathbf{W}^1, \mathbf{W}^2)$

  is ∞-quasi continuous. (Aida '11)

- $\mathcal{L}$ and its domain are compatible with constant multiplication and CM-shift

Rate function

Set a good rate function

$I_1 : G\Omega^B_{\alpha,4m}(\mathbb{R}^n) \to [0,\infty]$ **by**

$$I_1(\mathrm{w}) = \begin{cases} \dfrac{\|h\|^2_{\mathcal{H}}}{2} & (\text{if } \mathrm{w} = \mathcal{L}(h) \text{ for } \exists h \in \mathcal{M}^{x,a}), \\ \infty & (\text{otherwise}). \end{cases}$$

Also set

$$\hat{I}_1(\mathrm{w}) = I_1(\mathrm{w}) - \min\{\frac{\|h\|^2_{\mathcal{H}}}{2} \mid h \in \mathcal{M}^{x,a}\}$$

4 Main Result

[Theorem 1] Assume (A1), (A2) and the condition for $(\alpha, 4m)$. Then, we have

(1) The family $\{(\varepsilon\mathcal{L})_*[\delta_a(Y_1^\varepsilon)]\}_{\varepsilon>0}$ of finite measures satisfies an LDP on $G\Omega^B_{\alpha,4m}(\mathbb{R}^d)$ as $\varepsilon \searrow 0$ with a good rate function I_1

(2) Normalized meausures of the above satisfies an LDP with a good rate function $\hat{I}_1$.
(immediate from (1))

[Remark] Theorem 1 above also holds w.r.t. α'-Hölder geometric rough path topology for any $\alpha' \in (1/3, 1/2)$, since we can find α, m with that condition such that $(\alpha, 4m)$-Besov topology is stronger than α'-Hölder topology.

[Remark] The contraction principle: For any continuous map F from the geometric rough path space to a Hausdorff topological space, the image measure $F_*(\varepsilon\mathcal{L})_*[\delta_a(Y_1^\varepsilon)]$ satisfies an LDP, too.
$\rightarrow$ You can take another (Lyons-)Itô map !

[Remark] In Theorem 1, "strongly hypoelliptic" canNOT be weakened to "hypoelliptic."
($\because$ $\exists$ a simple counterexample.)

[Remark] The drift is of a special form $\varepsilon^2 V_0(x)$.
We probably cannot extend Theorem 1 for $V_0(\varepsilon, x)$, unless $\lim_{\varepsilon \searrow 0} V_0(\varepsilon, x) \equiv 0$.

($\because$ This guess is based on a bad example of short time asymptotics of heat kernel in BenArous-Léandre [PTRF '91, "Part II"])

5 Corollaries

$A_i : \mathbb{R}^N \to \mathbb{R}^N$ vector fields which satisfies (A1) $(0 \le i \le d)$.

♠ Another SDE:

$$dZ_t^\varepsilon = \varepsilon \sum_{i=1}^{d} A_i(Z_t^\varepsilon) \circ dw_t^i + \varepsilon^2 A_0(Z_t^\varepsilon)dt$$

with $\quad Z_0^\varepsilon = z \in \mathbb{R}^N$.

♠ Skelton ODE: For $h \in \mathcal{H}$,

$$d\zeta_t = \sum_{i=1}^{d} A_i(\zeta_t)dh_t^i, \qquad \zeta_0 = z.$$

$\tilde{Z}^\varepsilon = \tilde{Z}^\varepsilon(\,\cdot\,, z, w)$; ∞-quasi conti. modification of

$$\mathcal{W} \ni w \mapsto Z^\varepsilon(\,\cdot\,, z, w) \in C^{\alpha-H}([0,1], \mathbb{R}^N)$$

$(1/3 < \alpha < 1/2)$.

$\longrightarrow \tilde{Z}^\varepsilon_*[\delta_a(Y_1^\varepsilon)] = \tilde{Z}^\varepsilon(\,\cdot\,, z, \,\cdot\,)_*[\delta_a(Y_1^\varepsilon)]$
well-defined as a measure on $C^{\alpha-H}([0,1], \mathbb{R}^N)$.

Rate functions

$$I_2, \hat{I}_2 : C^{\alpha-H}([0,1], \mathbb{R}^N) \to [0, \infty].$$

Set

$$I_2(b) := \begin{cases} \inf\{\frac{\|h\|_{\mathcal{H}}^2}{2} \mid h \in \mathcal{M}^{x,a} \text{ s.t. } b = \zeta(\,\cdot\,, z, h)\ \}, \\ \infty, \ (h \in \mathcal{M}^{x,a} \text{ s.t. } b = \zeta(\,\cdot\,, z, h)). \end{cases}$$

and $\hat{I}_2(b) := I_2(b) - \min\{\frac{\|h\|_{\mathcal{H}}^2}{2} \mid h \in \mathcal{M}^{x,a}\}.$

[Corollary 2] (Thm 2.1, Takanobu-Watanabe '93)

Let $1/3 < \alpha < 1/2$. Assume (A1), (A2) for V_i's and (A1) for A_i's. Then, we have

(1) The family $\{ \tilde{Z}^{\varepsilon}_{*}[\delta_a(Y^{\varepsilon}_1)] \}_{\varepsilon>0}$ satisfies an LDP on $C^{\alpha-H}([0,1],\mathbb{R}^N)$ as $\varepsilon \searrow 0$ with a good rate function I_2.

(2) Normalized meausures of the above satisfies an LDP with a good rate function $\hat{I}_2$.

• Special case

Take $\mathbb{R}^n = \mathbb{R}^l = \mathbb{R}^N$, $x = z$, $V_i = A_i$ for all i.
Write $a = x' \in \mathbb{R}^n$.

$$\Longrightarrow \quad X_t^\varepsilon = Y_t^\varepsilon = Z_t^\varepsilon, \quad \phi_t = \psi_t = \zeta_t$$
and $\mathcal{M}^{x,a} = \mathcal{K}^{x,x'}$.

Normalization of $\tilde{Z}_*^\varepsilon[\delta_a(Y_1^\varepsilon)]$ is nothing but the the pinned diffusion measure $Q_{x,x'}^\varepsilon$ associated with the generator $\varepsilon^2(V_0 + \frac{1}{2}\sum_{i=1}^d V_i^2)$ with the starting point x and the ending point x'.

[Corollary 3] (FW-type LDP for pinned diffusions)

Let $1/3 < \alpha < 1/2$ and assume (A1), (A2).

$\Longrightarrow$ **The family $\{Q^{\varepsilon}_{x,x'}\}_{\varepsilon>0}$ satisfies an LDP on $C^{\alpha-H}([0,1],\mathbb{R}^N)$ as $\varepsilon \searrow 0$.**

[Remark]

♣ **Even Corollary 3 looks new.**

♣ **However, there is a parallel result on compact manifolds. Analytic method + a bit of RP theory. Bailleul(-Mesnager-Norris) '13+, '14+**

6 Sketch of Proof of "Theorem 1, (1)"

Difficulty of proof (when compared to I. '12+).

Lower est. $>$ Upper est.

- **Upper estimate is similar to the one in I. '12+**
- **In lower estimate, it becomes difficult to prove non-degeneracy of deterministic Malliavin covariance (because it does fail at some CM paths in the hypoelliptic case)**

Keys in Upper Estimate

♠ **Integration by parts for Watanabe distributions**

$\longrightarrow$ **The generalized $\mathbb{E}$ morphs into the usual $\mathbb{E}$.**

♠ **Kusuoka-Stroock's quantitative proof of non-degeneracy of Malliavin covariance matrix**

$$\|(\det \sigma_{X_1^\varepsilon})^{-1}\|_{L^p} \leq K_1(p)\varepsilon^{-K_2} \qquad (\varepsilon \searrow 0)$$

♠ Using them, we get for $\forall \mathrm{w} \in G\Omega^B_{\alpha,4m}(\mathbb{R}^d)$,

$$\lim_{r \searrow 0} \overline{\lim_{\varepsilon \searrow 0}} \varepsilon^2 \log(\varepsilon \mathcal{L})_*[\delta_a(Y_1^\varepsilon)](B_r(\mathrm{w})) \le -I(\mathrm{w})$$

where $B_r(\mathrm{w})$ is the "ball" of radius r centered at w.
$\Longrightarrow$ upper estimates for compact subsets.

♠ For closed sets, we need large deviation estimate on RP space w.r.t. Gaussian capacities.
(shown in I. '12+)
[plus, compactness of embedding $G\Omega^B_{\alpha',4m} \hookrightarrow G\Omega^B_{\alpha,4m}$ if $\alpha' > \alpha$]

Keys in Lower Estimate

♠ **Non-degeneracy of deterministic Malliavin covariance. (Note: it fails at some $h \in \mathcal{H}$).**

[Lemma] Assume (A1), (A2). Let $x, x' \in \mathbb{R}^n$ and $h \in \mathcal{K}^{x,x'} := \{h \in \mathcal{H} \mid \phi(1, x, h) = x'\}$.
Then, for any $\varepsilon > 0$, there exists $h^\varepsilon \in \mathcal{K}^{x,x'}$ s.t.

(1) $\|h - h^\varepsilon\|_{\mathcal{H}} < \varepsilon$,

(2) $\sigma_{\phi_1}(h^\varepsilon)$ **is non-degenerate,**

(3) $\langle h, \bullet \rangle_{\mathcal{H}} \in \mathcal{W}^*$ **($\mathcal{W} :=$ Wiener sp.)**

Proof is done by hand and fairly long.

This breaks down if $\exists$ a drift term in skeleton ODE.

♠ **A modified version of Watanabe's asymptotic expansion theorem (in TW '93).**

- **We use it**

- for $X^{\varepsilon}(1, x, w + (h/\varepsilon))$ or $Y^{\varepsilon}(1, x, w + (h/\varepsilon))$

- at $h \in \mathcal{K}^{x,x'} \subset \mathcal{M}^{x,a}$ (if $\Pi_{\mathcal{V}} x' = a$)

  as in the prevous Lemma.

- **This version fits extremely well with localization procedure on RP space.**

The END