KPZ equation with fractional derivatives of white noise

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Background

- We consider "ω-wise" solutions of non-liner stochastic PDEs.
- Examples:

$$\partial_t h(t,x) = \Delta h(t,x) + |\nabla h(t,x)|^2 + \xi(t,x) : \text{KPZ}$$
$$\partial_t \Phi(t,x) = \Delta \Phi(t,x) - \Phi(t,x)^3 + \xi(t,x) : \Phi^4\text{-model}$$
$$x \in \mathbb{R}^d, t \geq 0$$

ξ : space-time white noise on $[0,\infty) \times \mathbb{R}^d$.

- The solution of stochastic heat equation:

$$\partial_t X(t,x) = \Delta X(t,x) + \xi(t,x)$$

satisfies $X(t,\cdot) \in C^{\frac{2-d}{2}-}$.

- We cannot define above non-liner terms, because they are related to the product $\xi\eta$ of $\xi \in C^\alpha$ and $\eta \in C^\beta$ with $\alpha + \beta \leq 0$.

Subcriticality

- $h(t,x) \to h_\delta(t,x) = \delta^{-\frac{2-d}{2}} h(\delta^2 t, \delta x)$,
  $\xi(t,x) \to \xi_\delta(t,x) = \delta^{\frac{d+2}{2}} \xi(\delta^2 t, \delta x)$ $(\delta > 0)$.
- Changing variables

$$\partial_t h(t,x) = \Delta h(t,x) + |\nabla h(t,x)|^2 + \xi(t,x)$$
$$\to \partial_t h_\delta(t,x) = \Delta h_\delta(t,x) + \delta^{\frac{2-d}{2}} |\nabla h_\delta(t,x)|^2 + \xi_\delta(t,x)$$

- Formally, the non-liner term vanish as $\delta \to 0$, iff $d = 1$: "subcritical"

Hairer's theory

- M. Haier introduced "regularity structure(RS)". (2013)
- We can construct RS from given stochastic PDE iff it is subcritical.
- We can define the "renormalization map" on RS, and obtain a result as below:

Theorem (Hairer, 2013)

Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a smooth, nonnegative, symmetric, and compactly supported function s.t. $\int \rho = 1$. Set $\rho_\epsilon(t,x) = \epsilon^{-3}\rho(\epsilon^{-2}t, \epsilon^{-1}x)$, and $\xi_\epsilon = \xi * \rho_\epsilon$.
Then, there exists a sequence of constants $\{C_\epsilon \sim \frac{1}{\epsilon}\}$ s.t. the sequence of solutions h_ϵ (local in time) to

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \xi_\epsilon(t,x), \quad x \in \mathbb{T}, t \geq 0$$

converges to a stochastic process h.

More rough noise

- Define $\partial_x^\gamma \xi = (-\partial_x^2)^{\frac{\gamma}{2}} \xi$ ($\gamma > 0$) as a random distribution s.t.

$$(-\partial_x^2)^{\frac{\gamma}{2}} \xi(\phi) = \xi((-\partial_x^2)^{\frac{\gamma}{2}} \phi) \quad (\forall \phi \in C_0^\infty(\mathbb{R}^2)),$$

  where $(-\partial_x^2)^{\frac{\gamma}{2}} \phi = \mathcal{F}^{-1}(|\xi|^\gamma \mathcal{F}\phi)$.
- Does it hold similar renormalization to the following equation?

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + (\partial_x h(t,x))^2 + \partial_x^\gamma \xi(t,x)$$

Main result

- This eq is subcritical iff $\gamma < \frac{1}{2}$.
- But similar renormalization holds iff $\gamma < \frac{1}{4}$.

Theorem (Hoshino, 2015)

Let $0 \leq \gamma < \frac{1}{4}$. Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a smooth, nonnegative, symmetric, and compactly supported function such that $\int \rho = 1$.
Then, there exists a sequence of constants $\{C_\epsilon \sim \epsilon^{-1-2\gamma}\}$ s.t. the sequence of solutions h_ϵ (local in time) to

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \partial_x^\gamma \xi_\epsilon(t,x), \qquad x \in \mathbb{T}, t \geq 0$$

converges to a stochastic process h.

Notations

- $|k|_{\mathfrak{s}} := 2k_0 + k_1$, $\partial^k := \partial_t^{k_0}\partial_x^{k_1}$ $(k = (k_0, k_1) \in \mathbb{Z}_+^2)$.
- $||z||_{\mathfrak{s}} := \sqrt{|t|} + |x|$ $(z = (t, x) \in \mathbb{R}^2)$.
- For a function ρ on $\mathbb{R}^2$,

$$\mathcal{S}_{\mathfrak{s},z}^{\delta}\rho(z') := \delta^{-3}\rho(\delta^{-2}(t'-t), \delta^{-1}(x'-x)).$$

- $\mathcal{B}_r = \{\rho \in C_0^\infty(\mathbb{R}^2); ||\rho||_{C^r} \le 1, \mathrm{supp}\rho \in B_{\mathfrak{s}}(0,1)\}$ $(r \in \mathbb{N})$.
- $\mathcal{C}_{\mathfrak{s}}^{\alpha}(\mathbb{R}^2)$ $(\alpha > 0)$: locally α-Hölder space w.r.t $||\cdot||_{\mathfrak{s}}$.
- $\mathcal{C}_{\mathfrak{s}}^{\alpha}(\mathbb{R}^2)$ $(\alpha < 0)$: All of distributions ξ s.t.

$$|\xi(\mathcal{S}_{\mathfrak{s},z}^{\delta}\rho)| \lesssim \delta^{\alpha},$$

  uniformly over $\rho \in \mathcal{B}_r$ $(r = \lceil -\alpha \rceil)$ and locally uniformly over $z \in \mathbb{R}^2$.

Abstraction of stochastic PDE

- Mild form

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \eta \ (\eta : \text{smooth noise}), h(0,\cdot) = h_0$$
$$\Leftrightarrow h = G * (1_{t>0}((\partial_x h)^2 + \eta)) + Gh_0.$$

- Formally, h is represented by a sum of distributions with negative regularities, and a remainder.
- We reformulate KPZ eq into an equation of $H = \sum_\alpha h_\alpha \tau_\alpha$ (τ_α : basis vector, h_α : a function on $\mathbb{R}^2$) which values in an abstract liner space.

$$H = \mathcal{G}(1_{t>0}((\partial H)^2 + \Xi)) + Gh_0$$

Regularity structures

Definition (Regularity structure)

A regularity structure (A, T, G) consists of the following elements:

- $A \subset \mathbb{R}$, locally finite, bounded from below, $0 \in A$.
- $T = \bigoplus_{\alpha \in A} T_\alpha$. Each T_α is a liner space with a norm $||\cdot||_\alpha$.
- G is a group of linear operators $T \to T$, such that

$$\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha, \beta\in A} T_\beta$$

  for all $\Gamma \in G$, $\alpha \in A$, $\tau \in T_\alpha$.
- $\alpha_0 := \inf A$.

Model

Definition (Model)

A model (Π, Γ) on $\mathbb{R}^d$ for a regularity structure (A, T, G) consists of the following elements:

- $\Pi_z (z \in \mathbb{R}^d) : T \to \mathcal{D}'(\mathbb{R}^d)$ is linear map such that

$$|(\Pi_z \tau)(\mathcal{S}^\delta_{\mathfrak{s},z} \rho)| \lesssim \delta^\alpha \text{ (loc in } z),$$

  for all $\alpha \in A$, $\tau \in T_\alpha$, and $\rho \in \mathcal{B}_r$ $(r = \lceil -\alpha_0 \rceil)$.
- $\Gamma_{z,z'} (z, z' \in \mathbb{R}^d) \in G$ is an element such that

$$||\Gamma_{z,z'} \tau||_\beta \lesssim ||z - z'||_{\mathfrak{s}}^{\alpha-\beta} \text{ (loc in } z, z'),$$

  for all $\alpha, \beta \in A$ $(\beta < \alpha)$ and $\tau \in T_\alpha$.
- $\Pi_z \Gamma_{z,z'} = \Pi_{z'}$, $\Gamma_{z,z'} \Gamma_{z',z''} = \Gamma_{z,z''}$ $(\forall z, z', z'' \in \mathbb{R}^d)$.

Reconstruction

Definition (Modelled distribution)

Let $(\mathbf{\Pi}, \mathbf{\Gamma})$ be a model. $f : \mathbb{R}^d \to T$ is in $\mathcal{D}^\gamma$ ($\gamma \in \mathbb{R}$) iff

$$||f(z) - \Gamma_{z,z'} f(z')||_\alpha \lesssim ||z - z'||_{\mathfrak{s}}^{\gamma-\alpha} \text{ (loc in } z, z'),$$

for all $\alpha \in A$ with $\alpha < \gamma$.

Theorem (Reconstruction theorem)

Let $(\mathbf{\Pi}, \mathbf{\Gamma})$ *be a model and* $\gamma > 0$. *Then there exists a unique liner map* $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{C}_{\mathfrak{s}}^{\alpha_0}(\mathbb{R}^d)$ *s.t.*

$$|(\mathcal{R}f - \Pi_z f(z))(\mathcal{S}_{\mathfrak{s},z}^{\delta} \rho)| \lesssim \delta^\gamma \text{ (loc in } z),$$

for all $f \in \mathcal{D}^\gamma$.

RS for KPZ

- $\mathcal{U}$:All of basis vectors that describe h.
  $\mathcal{V}$:All of basis vectors that describe $(\partial_x h)^2 + \eta$.

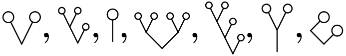
$$
\begin{aligned}
&1, X_0, X_1, \Xi \in \mathcal{V},\\
&\tau \in \mathcal{V} \Rightarrow \mathcal{I}\tau \in \mathcal{U} \ (\mathcal{I}(X^k) = 0),\\
&\partial(X_0^{k_0} X_1^{k_1}) = k_1 X_0^{k_0} X_1^{k_1-1},\\
&\tau_1, \tau_2 \in \mathcal{U} \Rightarrow \partial\tau_1 \partial\tau_2 \in \mathcal{V}.
\end{aligned}
$$

- Regularities of basis vectors.

$$
\begin{aligned}
&|1| = 0, |X_0| = 2, |X_1| = 1, |\Xi| = \alpha_0 \ \text{(to be determined)}\\
&|\tau\tau'| = |\tau| + |\tau'|\\
&|\mathcal{I}\tau| = |\tau| + 2, \ |\partial\mathcal{I}\tau| = |\tau| + 1
\end{aligned}
$$

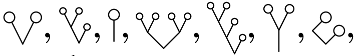
- $\mathcal{F} = \mathcal{U} \cup \mathcal{V}$, $T = \mathrm{span}\mathcal{F}$.

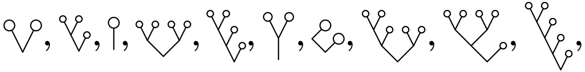
RS for KPZ

- $\xi_\epsilon \to \xi$ in prob in $\mathcal{C}_{\mathfrak{s}}^{-\frac{3}{2}-\kappa}$ $(\kappa > 0)$.
- We use shorthand notations to describe basis vectors.
- We should choose $\alpha_0 = -\frac{3}{2} - \kappa$ ($\kappa > 0$: sufficiently small)
  $\Rightarrow \Xi,$  $, \mathbf{1}, \cdots \in \mathcal{F}$

RS for fractional case

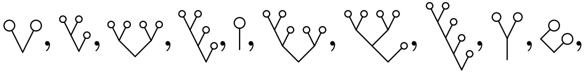
- $\partial_x^\gamma \xi_\epsilon \to \partial_x^\gamma \xi$ in prob in $\mathcal{C}_{\mathfrak{s}}^{-\frac{3}{2}-\gamma-\kappa}$ $(\kappa > 0)$.
- We should choose $\alpha_0 = -\frac{3}{2} - \gamma - \kappa$ ($\kappa > 0$: sufficiently small)
- $0 \leq \gamma < \frac{1}{10}$

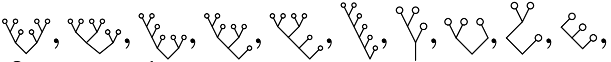
  $\Rightarrow \Xi,$  $, \mathbf{1}, \dots$
- $\frac{1}{10} \leq \gamma < \frac{1}{6}$

  $\Rightarrow \Xi,$  $, \mathbf{1}, \dots$

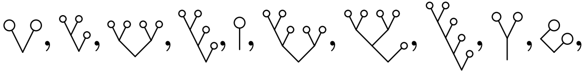
RS for fractional case

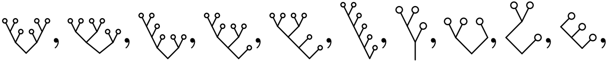
- $\frac{1}{6} \leq \gamma < \frac{3}{14}$

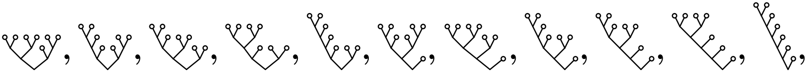
  $\Rightarrow \boldsymbol{\Xi},$ 

   $\mathbf{1}, \ldots$

- $\frac{3}{14} \leq \gamma < \frac{1}{4}$

  $\Rightarrow \boldsymbol{\Xi},$ 

   $\mathbf{1}, \ldots$

- and so on...

Canonical model

- There is a canonical lift η into $(\mathbf{\Pi}^{\eta}, \mathbf{\Gamma}^{\eta})$:

$$\Pi_z \mathbf{1}(z') = 1, \ \Pi_z \Xi(z') = \eta(z') \tag{1}$$

$$\Pi_z X_0(z') = t' - t, \ \Pi_z X_1(z') = x' - x \tag{2}$$

$$\Pi_z \tau\tau' = (\Pi_z \tau)(\Pi_z \tau') \tag{3}$$

$$\Pi_z \mathcal{I}\tau = G * \Pi_z \tau - \sum_{|k| < |\mathcal{I}\tau|} \frac{(\cdot - z)^k}{k!} \partial^k G * \Pi_z \tau \tag{4}$$

$$\Pi_z \partial\mathcal{I}\tau = \partial_x G * \Pi_z \tau - \sum_{|k| < |\partial\mathcal{I}\tau|} \frac{(\cdot - z)^k}{k!} \partial^k \partial_x G * \Pi_z \tau \tag{5}$$

Admissible model

- Assume (1)(2)(4)(5) $\to$ "Admissible model"
- For each admissible model Z, a linear map $\mathcal{G} : \mathcal{D}^\theta \to \mathcal{D}^{\theta+2}$ is defined and satisfies

$$\mathcal{R}\mathcal{G}f = G * \mathcal{R}f, \quad \mathcal{G} = \mathcal{I} + (\{X^k\}\text{-valued part})$$

Solution map

Theorem (Hairer, 2013)

- $\alpha_0 \in (-2, -\frac{3}{2})$, $\theta > -\alpha_0$, $\zeta \in (0, \alpha_0 + 2)$
- *For each initial condition h_0 and admissible model Z, there exists $T > 0$ s.t. the equation*

  $$H = \mathcal{G}(1_{t>0}((\partial H)^2 + \Xi)) + G h_0$$

  has a unique solution $H \in \mathcal{D}^{\theta,\zeta}$ (permit singularity as $t \to 0+$) on $t \in [0, T]$.
- $S : (h_0, Z) \mapsto H$ *is continuous.*

Solution map

- When $Z = (\Pi^\eta, \Gamma^\eta)$ for some smooth noise η, we have

$$\begin{aligned} \mathcal{R}H &= G * (1_{t>0}\mathcal{R}((\partial H)^2 + \Xi)) \\ &= G * (1_{t>0}((\partial_x \mathcal{R}H)^2 + \eta)). \end{aligned}$$

So $h = \mathcal{R}H$ solves $\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \eta$.

Renormalized KPZ eq

- $\alpha_0 = -\frac{3}{2} - \kappa$ ($\kappa > 0$: sufficiently small)
- For any constants $C_{\;}, C_{\;}, C_{\;}$, the set of functions

$$\widehat{\Pi}^\eta_z \tau = \Pi^\eta_z \tau \ (\tau = \Xi, \;, \;, \;, \;, 1),$$

$$\widehat{\Pi}^\eta_z \; = \Pi^\eta_z \; - C_{\;},$$

$$\widehat{\Pi}^\eta_z \; = \Pi^\eta_z \; - C_{\;},$$

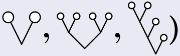
$$\widehat{\Pi}^\eta_z \; = \Pi^\eta_z \; - C_{\;}$$

  is uniquely extended to an admissible model $(\widehat{\Pi}^\eta, \widehat{\Gamma}^\eta)$.
- $h = S(h_0, \widehat{Z}^\eta)$ solves the equation

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 - (C_{\;} + C_{\;} + 4C_{\;}) + \eta$$

Convergence of renormalized models

Theorem

- *Choose proper* $C_\tau^{(\epsilon)}$ $(\tau =$  $)$.
- *Let* $Z^{(\epsilon)}$ *be a model lifted from* ξ_ϵ
- *Let* $\widehat{Z}^{(\epsilon)}$ *be the renormalized model. Then there exists an admissible random model* $\widehat{Z}$ *s.t.*

$$\widehat{Z}^{(\epsilon)} \to \widehat{Z} \text{ in prob } \ (\epsilon \to 0).$$

For the fractional case

Theorem

- *Let* $\mathbf{0} \leq \gamma < \frac{1}{4}$
- *Choose proper* $C_{\tau}^{(\epsilon)}$ *for all* $\tau \in \mathcal{F}$ *with* $\|\tau\| = 2, 4, 6$.
- *Let* $Z^{(\epsilon)}$ *be a model lifted from* $\partial_x^{\gamma} \xi_{\epsilon}$
- *Let* $\widehat{Z}^{(\epsilon)}$ *be the renormalized model. Then there exists an admissible random model* $\widehat{Z}$ *s.t.*

$$\widehat{Z}^{(\epsilon)} \to \widehat{Z} \text{ in prob } (\epsilon \to 0).$$

Proof of renormalization

- $\widehat{\Pi}_0^{(\epsilon)}\tau(\mathcal{S}_{\mathfrak{s},0}^{\lambda}\phi)$ convergences for all $|\tau| \le 0$
  $\Rightarrow (\widehat{\Pi}^{(\epsilon)}, \widehat{\Gamma}^{(\epsilon)})$ convergences.
- For each $\tau \in \mathcal{F}$, $\widehat{\Pi}_0^{(\epsilon)}\tau$ has the form

$$\widehat{\Pi}_0^{(\epsilon)}\tau(z) = \sum I_k(\widehat{\mathcal{W}}^{(\epsilon,k)}\tau(z;\cdot,\dots,\cdot))$$

  I_k:kth multiple Wiener-Itô integral. $\widehat{\mathcal{W}}^{(\epsilon,k)}\tau(z) \in (L^2(\mathbb{R}\times\mathbb{T}))^{\otimes k}$.
- By Itô isometry, we have

$$\begin{aligned}
&\mathbb{E}|\widehat{\Pi}_0^{(\epsilon)}\tau(\mathcal{S}_{\mathfrak{s},0}^{\lambda}\phi)| \\
&\lesssim \sum \int \mathcal{S}_{\mathfrak{s},0}^{\lambda}\phi(z)\mathcal{S}_{\mathfrak{s},0}^{\lambda}\phi(z') \\
&\qquad\qquad \times \langle \widehat{\mathcal{W}}^{(\epsilon,k)}\tau(z), \widehat{\mathcal{W}}^{(\epsilon,k)}\tau(z')\rangle_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes k}} dz dz'
\end{aligned}$$

Proof of renormalization

- Assume that there exist $\widehat{\mathcal{W}}^{(k)}\tau(z)$ s.t.

$$|\langle \widehat{\mathcal{W}}^{(k)}\tau(z), \widehat{\mathcal{W}}^{(k)}\tau(z')\rangle_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes k}}| \lesssim \|z-z'\|_{\mathfrak{s}}^{2|\tau|+\delta}$$
$$|\langle \delta\widehat{\mathcal{W}}^{(\epsilon,k)}\tau(z), \delta\widehat{\mathcal{W}}^{(\epsilon,k)}\tau(z')\rangle_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes k}}| \lesssim \epsilon^{\delta}\|z-z'\|_{\mathfrak{s}}^{2|\tau|+\delta}$$
$$(\delta\mathcal{W}^{(\epsilon,k)} = \mathcal{W}^{(k)} - \mathcal{W}^{(\epsilon,k)})$$

  with $2|\tau| > -3$, for some $\delta > 0$.
- Note that

$$\int \mathcal{S}_{\mathfrak{s},0}^{\lambda}\phi(z)\mathcal{S}_{\mathfrak{s},0}^{\lambda}\phi(z')\|z-z'\|_{\mathfrak{s}}^{2|\tau|+\delta}dzdz' \lesssim \lambda^{2|\tau|+\delta}$$

- Then $\widehat{Z}^{(\epsilon)} \to \widehat{Z}$.

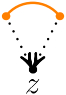
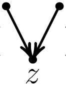
Proof of renormalization

- $\widehat{\Pi}_0^{(\epsilon)}\text{⫯}(z) = \partial_x G * \partial_x^\gamma \xi(z) = I_1(\partial_x^\gamma \partial_x G * \rho_\epsilon(z - \cdot))$
- $\widehat{\mathcal{W}}^{(\epsilon,1)}\text{⫯}(z;\cdot) = \partial_x^\gamma \partial_x G * \rho_\epsilon(z - \cdot)$
- $\widehat{\mathcal{W}}^{(1)}\text{⫯}(z;\cdot) = \partial_x^\gamma \partial_x G(z - \cdot)$
- $|\partial_x^\gamma \partial_x G(z)| \lesssim ||z||_{\mathfrak{s}}^{-2-\gamma}$ around $\mathbf{0}$.
- $\langle \widehat{\mathcal{W}}^{(1)}\text{⫯}(z), \widehat{\mathcal{W}}^{(1)}\text{⫯}(z') \rangle_{(L^2(\mathbb{R}\times\mathbb{T}))}$:

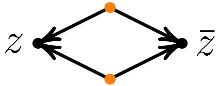
$\lesssim ||z - z'||_{\mathfrak{s}}^{-1-2\gamma}$

$-1 - 2\gamma > 2|\text{⫯}|$.

Proof of renormalization

- $\Pi_0^{(\epsilon)}\overset{\circ\,\circ}{\vee}(z) = I_2($  $) +$ 
- $\widehat{\Pi}_0^{(\epsilon)}\overset{\circ\,\circ}{\vee}(z) = I_2($  $)$, $C^{(\epsilon)}_{\overset{\circ\,\circ}{\vee}} =$ 
- $\widehat{\Pi}_0\overset{\circ\,\circ}{\vee}(z) = I_2($  $)$

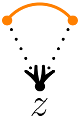
- $\langle \widehat{\mathcal{W}}^{(2)}(z), \widehat{\mathcal{W}}^{(2)}(z') \rangle_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes 2}}$:

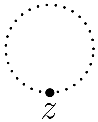


$\lesssim \|z - z'\|_{\mathfrak{s}}^{-2-4\gamma}$

$-2 - 4\gamma > 2| \,|$.

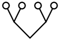
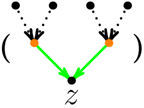
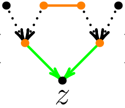
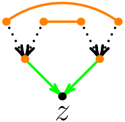
- $-2 - 4\gamma > -3 \Leftrightarrow \gamma < \frac{1}{4}$.

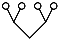
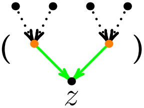
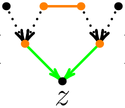
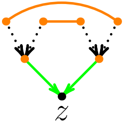
- 

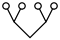
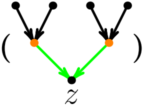
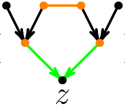
$\lesssim$


$\lesssim \epsilon^{-1-2\gamma}$

Proof of renormalization

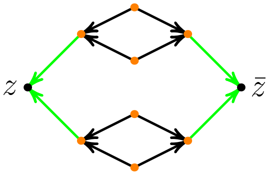
- $\mathbf{\Pi}_0^{(\epsilon)}$$(z) = I_4($  $) + I_2($  $) +$ 

- $\widehat{\mathbf{\Pi}}_0^{(\epsilon)}$$(z) = I_4($  $) + I_2($  $)$, C$^{(\epsilon)} =$ 

- $\widehat{\mathbf{\Pi}}_0^{(\epsilon)}$$(z) = I_4($  $) + I_2($  $)$,

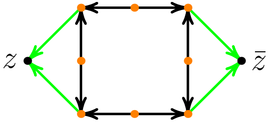
Proof of renormalization

- $\langle \widehat{\mathcal{W}}^{(4)}(z), \widehat{\mathcal{W}}^{(4)}(z') \rangle_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes 4}}$:

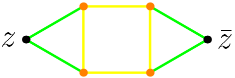


$\lesssim \|z - z'\|_{\mathfrak{s}}^{-8\gamma}$

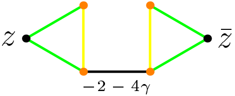
- $\langle \widehat{\mathcal{W}}^{(2)}(z), \widehat{\mathcal{W}}^{(2)}(z') \rangle_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes 2}}$:



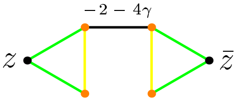
$\lesssim$



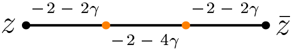
$\lesssim$



$+$

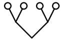
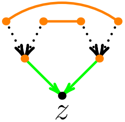


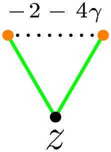
$\lesssim$



$\lesssim \|z - z'\|_{\mathfrak{s}}^{-8\gamma}$

Proof of renormalization

- $-8\gamma > 2|$$|$.
- 

$\lesssim$ 

$\lesssim \epsilon^{-4\gamma}$

References

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- M. Hairer. Singular stochastic PDEs. ArXiv:1403.6353