# KPZ EQUATION WITH FRACTIONAL DERIVATIVES OF WHITE NOISE

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We discuss the stochastic partial differential equation

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + (\partial_x h(t,x))^2 + \partial_x^\gamma \xi(t,x) \tag{1}$$

for $(t,x) \in [0,\infty) \times \mathbb{T}$ with $\gamma \geq 0$. Here, $\xi$ is a space-time white noise on $[0,\infty) \times \mathbb{T}$ and $\partial_x^\gamma = -(-\partial_x^2)^{\frac{\gamma}{2}}$ is the fractional derivative. When $\gamma = 0$, this equation is called KPZ equation, which is proposed in [3] as a model of surface growth. Hairer discussed the solvability of KPZ equation in [1]. He showed in [1] that the renormalized equation

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \xi_\epsilon(t,x), \tag{2}$$

where $\xi_\epsilon$ is a smooth approximation of $\xi$ and $C_\epsilon \sim \frac{1}{\epsilon}$ is a sequence of constants, has a unique limiting process $h$, which is independent of the way to approximate $\xi$.

Our goal is to make the noise rougher and see to what extent this theory works. Because of the "local subcriticality" ([2]), we can expect that the similar results hold if $\gamma < \frac{1}{2}$. However, we show that the renormalization like (2) is possible only for $0 \leq \gamma < \frac{1}{4}$.

**Theorem 1.** *Let $\rho = \rho(t,x)$ be a function on $\mathbb{R}^2$ which is smooth, compactly supported, symmetric in $x$, nonnegative, and satisfies $\int_{\mathbb{R}^2} \rho(t,x)dtdx = 1$. Let $0 \leq \gamma < \frac{1}{4}$ and $0 < \alpha < \frac{1}{2} - \gamma$. Then there exists a sequence of constants $C_\epsilon$ such that*

1. *We have $C_\epsilon \leq C\epsilon^{-1-2\gamma}$ for some constant $C$ (depending on $\gamma$ and $\rho$).*
2. *For every initial condition $h_0 \in \mathcal{C}^\alpha(\mathbb{T})$, the sequence of solutions $h_\epsilon$ to the equation:*
$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \partial_x^\gamma \xi_\epsilon(t,x)$$
*on $(t,x) \in [0,T) \times \mathbb{T}$ for some random time $T$, converges to a unique stochastic process $h$, which is independent of the choice of $\rho$.*

*This convergence holds in probability in the uniform norm on all compact sets in $[0,T) \times \mathbb{T}$ and $\alpha$-Hölder norm on all compact sets in $(0,T) \times \mathbb{T}$.*

## References


[1] Hairer, M.: Solving the KPZ equation, Ann. Math. (2), **178** (2013) 559–664.
[2] Hairer, M.: A theory of regularity structures, Invent. Math., **198** (2014) 269–504.
[3] Kardar, M., Parisi, G., and Zhang, Y.-C.: Dynamic scaling of growing interfaces, Phys. Rev. Lett., **56** (1986) 889–892.