Locality property and a related continuity problem for SLE and SKLE II

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October 21, 2015

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    - Generalized Itô formula applied to the image process $\widetilde{\xi}(t)$
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2. Some overview

Perturbations of the domain and a generalized Komatu Loewner equation for image hulls $\{\widetilde{F}_t\}$

Let $\alpha(\mathbf{s})$ and $b(\mathbf{s})$ be homogeneous functions on S of degree 0 and -1, respectively,
both satisfying the Lipschitz continuity condition **(L)**
Let $(\xi(t), \mathbf{s}(t)),\ t < \zeta$, be the unique strong solution of The stochastic differential equation

$$\xi(t) = \xi + \int_0^t \alpha(\mathbf{s}(t) - \widehat{\xi}(t))dB_s + \int_0^t b(\mathbf{s}(t) - \widehat{\xi}(t))ds \quad (1.1)$$

$$\mathbf{s}_j(t) = \mathbf{s}_j + \int_0^t b_j(\mathbf{s}(t) - \widehat{\xi}(t))ds, \quad t \geq 0, \quad 1 \leq j \leq 3N, \quad (1.2)$$

where ζ is the time that $(\xi(t), \mathbf{s}(t))$ approaches the point at infinity of $\mathbb{R} \times S$.
The coefficients $b_j,\ 1 \leq j \leq 2N$, are determined by the complex Poisson kernel. They are homogeneous with degree -1 and satisfy condition **(L)**.

We write $D_t = D(\mathbf{s}(t)) \in \mathcal{D},\ D = D_0$
and we substitute $(\xi(t), \mathbf{s}(t))$ into the K-L equation

$$\frac{d}{dt} g_t(z) = -2\pi \Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)), \ \text{with } g_0(z) = z \in D, \tag{1.3}$$

which admits a unique solution $g_t(z),\ t \in [0, t_z)$ passing through
$G = \bigcup_{t \in [0,\zeta)} \{t\} \times D_t$.

The associated family of random growing $\mathbb{H}$-hulls $F_t = \{z \in \mathbb{H} : t_z \leq t\}$
is denoted by $\mathrm{SKLE}_{\alpha,b}$
and is called a stochatic Komatu-Loewner evolution.

g_t is the canonical map from $D \setminus F_t$.

Let us consider any $\mathbb{H}$-hull $A \subset D(= D(\mathbf{s}(0)))$ and a canonical map Φ_A from $D \setminus A$ onto $\widetilde{D} \in \mathcal{D}$.

We let $\tau_A = \inf\{t > 0 : \overline{F}_t \cap \overline{A} \neq \emptyset\}$.

We only consider those parameters t with $t < \tau_A$.

Define the image hulls of $\{F_t\}$ by

$$\widetilde{F}_t = \Phi_A(F_t), \quad t < \tau,$$

Let $\widetilde{g}_t$ be the canonical map from $\widetilde{D} \setminus \widetilde{F}_t$ onto $\widetilde{D}_t$ and $\widetilde{a}_t$ be its half-plane capacity: $\widetilde{a}_t = \lim_{\widetilde{z} \to \infty} \widetilde{z}(\widetilde{g}_t(\widetilde{z}) - \widetilde{z})$.

Along with the canonical maps g_t, Φ_A and $\widetilde{g}_t$, we consider the canonical map h_t from $D_t \setminus g_t(A)$. Then

$$\widetilde{g}_t \circ \Phi_A = h_t \circ g_t \tag{1.4}$$

because both of them are canonical maps from $D \setminus (F_t \cup A)$.

See Figure 3.

Define

$$\widetilde{\xi}(t) = h_t(\xi(t)). \tag{1.5}$$

Denote by $\widetilde{\Psi}_t(z,x),\ z \in \widetilde{D}_t,\ x \in \partial\mathbb{H}$, the BMD-complex Poisson kernel of $\widetilde{D}_t$.

The derivative of a function f in the time parameter is designated by $\dot{f}$.

We can then prove the following generalized Komatu Loewner equation for the image hulls $\{\widetilde{F}_t\}$.

Theorem 1.1

(i) *It holds that*

$$\dot{\widetilde{a}}_s \ = \ 2|h_s'(\xi(s))|^2. \tag{1.6}$$

(ii) *For* $t \in (0,\tau)$ *and* $z \in \widetilde{D} \setminus \widetilde{F}_t$,
$\widetilde{g}_s(z)$ *is continuously differentiable in* $s \in [0.t]$ *and*

$$\frac{d\widetilde{g}_s(z)}{ds} = -2\pi|h_s'(\xi(s))|^2\ \widetilde{\Psi}_s(\widetilde{g}_s(z),\widetilde{\xi}(s)), \quad g_0(z) = z. \tag{1.7}$$

[The proof of Theorem 1.1 (i)]

The identity $\dot{\tilde{a}}(t) = 2|h_t'(\xi(t))|^2$ is well known for SLE_κ.

We use a comparison theorem of the half-plane capacity between simply and multiply connected domains first obtained by

[D] S. Drenning, Excursion reflected Brownian Motions and Loewner equations in multiply connected domains, arXiv:1112.4123, 2011

for Jordan arcs using ERBM.

This comparison (identification) theorem is extended to a general growing hulls using BMD in Appendix of

[CF3] Z.-Q. Chen and M. Fukushima, Stochastic Komatu-Loewner evolution and BMD domain constant, arXiv:1410.8257vl

The proof of Theorem 1.1 (ii) requires the following several steps:

Step I. Joint continuity of $\Im \widetilde{g}_t(z)$

We express $\Im \widetilde{g}_t(z)$ in terms of $\Im g_t(z)$ and the ABM on $\mathbb{H}$.

$\Im g_t(z)$ is jointly continuous in (t, z) as $g_t(z)$ is the solution of the ODE (1.3).
Hence the joint continuity of $\Im \widetilde{g}_t(z)$ follows from this expression.

The above expression is obtained by combining the relation

$$\widetilde{g}_t = h_t \circ g_t \circ \Phi_A^{-1}$$

with the probabilistic expression of the conformal map h_t in terms of the BMD on D_t found in [CFR]

and the conformal invariance of BMD and ABM.

Step II. Joint continuity of $\widetilde{g}_t(z)$

In a similar way to §7, 8 of [CFR], we can deduce from Step I that, for $t \in [0, \tau)$, $\widetilde{g}_s(\widetilde{z})$ is jointly continuous in

$$(s, \widetilde{z}) \in [0, t] \times [(\widetilde{D} \cup \partial\mathbb{H}) \setminus \overline{\widetilde{F}_t} \setminus \widetilde{A}].$$

Step III. Joint continuity of $h_t(z),\ h_t'(z),\ h_t''(z)$

which can be obtained by combining Step II with the relation

$$h_t = \widetilde{g}_t \circ \Phi_A \circ g_t^{-1}$$

and the continuity of the solution of the K-L equation (1.3) with respect to the initial time and initial position.

This step is also very crucial for an use of Itô formula in the next section.

Derivation of the generalized K-L equation (1.7)

$\{F_t\}$ is right continuous with limit $\xi(t)$ by Theorem 2.2 of Lecture 1. Hence $\{\widetilde{F}_t\}$ is right continuous with limit $\widetilde{\xi}(t)$.

From Step I, we can also deduce that $\lim_{t\downarrow s} \widetilde{g}_t \circ \widetilde{g}_s^{-1}(z) = z$ locally uniformly in z.

By combining those two properties, we can show that the equation (1.7) holds true in the right derivative sense.

Since the right hand side of this equation as well as $\widetilde{g}_t(z)$ are continuous in t by virtue of Steps II, III, we conclude that (1.7) is a genuine ODE.

Generalized Itô formula applied to the image process $\widetilde{\xi}(t)$

The BMD domain constant is a function on S defined by

$$b_{\rm BMD}(\mathbf{s}) = \lim_{z\to\mathbf{0}} 2\pi(\Psi_{D(\mathbf{s})}(z,\mathbf{0}) - \Psi^{\mathbb{H}}(z,\mathbf{0})), \quad \mathbf{s}\in S. \qquad (1.8)$$

It can be shown to be a homogeneous function on S of degree -1 satisfying the Lipschitz condition **(L)**
by a conformal invariance of BMD and the the Lipschitz continuity of the BMD complex Poisson kernel Ψ shown in [CFR].

The BMD domain constant indicates a descrepancy of the slit domain $D(\mathbf{s})$ from the upper half plane $\mathbb{H}$ with respect to BMD.

We put $b_{\rm BMD}(\xi,\mathbf{s}) = b_{\rm BMD}(\mathbf{s}-\widehat{\xi})$ for $\xi\in\mathbb{R}$, $\mathbf{s}\in S$.

Notice that $\widetilde{\xi}(t) = h_t(\xi(t))$ where $h_t(z)$ is extended to be a conformal map from

$$G_t = (D_t \cup \Pi D_t \cup \partial\mathbb{H}) \setminus (g_t(\overline{A}) \cup \Pi g_t(A)),$$

$h_t(z)$ is a random adapted process.
$\xi(t) \in \partial\mathbb{H}$ is a continuous semi-martingale by (1.1):

$$d\xi(t) = \alpha(\mathbf{s}(t) - \widehat{\xi}(t))dB_t + b(\mathbf{s}(t) - \widehat{\xi}(t))dt. \tag{1.9}$$

By step III in the above, $h_t'(z), h_t''(z)$ are jointly continuous.
If one can further check that

$$h_t(z) \text{ is differntiable in } t \text{ for each } z \in \partial\mathbb{H}\setminus\overline{A} \text{ and } \dot{h}_t(z) \text{ is jointly continuous,} \tag{1.10}$$

then, a generalized Itô formula (see §2 below) applies in getting

$$d\widetilde{\xi}(t) = \dot{h}_t(\xi(t))dt + h_t'(\xi(t))d\xi(t) + \frac{1}{2}h_t''(\xi(t))d\langle\xi\rangle_t. \tag{1.11}$$

It readily follows from the relation $h_t = \widetilde{g}_t \circ \Phi_A \circ g_t^{-1}$ and the generalized K-L equation (1.7) that, for $z \in D_t \setminus g_t(A)$,

$$\dot{h}_t(z) = -2\pi |h_t'(\xi(t))|^2 \widetilde{\Psi}_t(h_t(z), h_t(\xi(t))) + 2\pi h_t'(z) \Psi_{\mathbf{s}(t)}(z, \xi(t)), \tag{1.12}$$

Let $B \in \mathbb{C}$ be the disk centered at $\xi(t)$ with $\overline{B} \subset G_t$.
Expressing $(h_u(z) - f_t(z))/(u-t),\ z \in B$, by the Cauchy integral formula and letting $u \to t$ by taking Step III in the above into account, we can check the condition (1.10) and see that $\dot{h}_t(z)$ is analytic in $z \in B$. In particlular, $\dot{h}_t(\xi(t))$ can be computed by $\lim_{z\to\xi(t),\ z\in\mathbb{H}} \dot{h}_t(z)$ yielding

$$\begin{aligned}
\dot{h}_t(\xi(t)) &= h_t'(\xi(t))\, b_{\rm BMD}(\xi(t), \mathbf{s}(t)) - |h_t'(\xi(t))|^2\, b_{\rm BMD}(h_t(\xi(t)), h_t(\mathbf{s}(t))) \\
&\quad + \lim_{z\to\xi(t)} \left(\frac{2|h_t'(\xi(t))|^2}{h_t(z) - h_t(\xi(t))} - \frac{2h_t'(z)}{z - \xi(t)} \right) \\
&= h_t'(\xi(t))\, b_{\rm BMD}(\xi(t), \mathbf{s}(t)) - |h_t'(\xi(t))|^2\, b_{\rm BMD}(h_t(\xi(t)), h_t(\mathbf{s}(t))) \\
&\quad - 3h_t''(\xi(t)).
\end{aligned} \tag{1.13}$$

We thus get from (1.9), (1.11) and (1.13)

Theorem 1.2

The image processs $\widetilde{\xi}(t)$ is a semi-martingale expreessed as

$$
\begin{aligned}
d\widetilde{\xi}(t) &= h_t'(\xi(t))\left(b(\mathbf{s}(t)-\widehat{\xi}(t)) + b_{\mathrm{BMD}}(\xi(t),\mathbf{s}(t))\right)dt \\
&+\frac{1}{2}h_t''(\xi(t))\left(\alpha(\mathbf{s}(t)-\widehat{\xi}(t))^2-6\right)dt \qquad (1.14) \\
&-|h_t'(\xi(t))|^2 b_{\mathrm{BMD}}(\widetilde{\xi}(t),h_t(\mathbf{s}(t)))dt + h_t'(\xi(t))\alpha(\mathbf{s}(t)-\widehat{\xi}(t))dB_t.
\end{aligned}
$$

Characterization of locality of $\mathrm{SKLE}_{\alpha,-b_{\mathrm{BMD}}}$

Let $\{\check{F}_t\}_{t<\check{\tau}}$ be the half-plane capacity reparametrization of the image hulls $\{\widetilde{F}_t\}_{t<\tau}$; namely,

$$\check{F}_t = \widetilde{F}_{\widetilde{a}^{-1}(2t)}, \qquad \check{\tau}_A = \widetilde{a}(\tau_A)/2. \tag{1.15}$$

where $\widetilde{a}(t)$ is the half-plane capacity of $\widetilde{F}_t$ and $\widetilde{a}^{-1}$ is its inverse function. Accordingly, the processes $\widetilde{\xi}(t) = h_t(\xi(t)) = \widetilde{g}_t \circ \Phi_A(\xi)$ and $\widetilde{\mathbf{s}}_j(t) = h_t(\mathbf{s}_j(t)) = \widetilde{g}_t \circ \Phi_A(\mathbf{s}_j)$ are time-changed into

$$\check{\xi}(t) = \widetilde{\xi}(\widetilde{a}^{-1}(2t)) \quad \text{and} \quad \check{\mathbf{s}}_j(t) = \widetilde{\mathbf{s}}_j(\widetilde{a}^{-1}(2t)), \quad 1 \le j \le 3N, \quad t < \check{\tau}. \tag{1.16}$$

Set $\check{g}_t = \widetilde{g}_{\widetilde{a}^{-1}(2t)}$ and $\check{\Psi}_t = \widetilde{\Psi}_{\widetilde{a}^{-1}(2t)}$.
It follows from (1.6), (1.7) and the joint continuity of $h_t'(z)$ that, for $T \in (0, \check{\tau})$, $\check{g}_t(z)$ is continuously differentiable in $t \in [0, T]$ and

$$\frac{d\check{g}_t(z)}{dt} = -2\pi \check{\Psi}_t(\check{g}_t(z), \check{\xi}(t)), \quad \check{g}_0(z) = z \in \widetilde{D} \setminus \check{F}_t. \tag{1.17}$$

Just as the K-L slit equation for $\mathbf{s}(t)$ follows from the Komatu-Loewner equation,
the following equation for $\check{\mathbf{s}}(t)$ follows from the equation (1.17).

Lemma 1.3

It holds under $\mathbb{P}_{(\xi,\mathbf{s})}$ *that*

$$\check{\mathbf{s}}_j(t) = \Phi_A(\mathbf{s}_j) + \int_0^t \check{b}_j(\check{\mathbf{s}}(s) - \widehat{\check{\xi}}(t))ds, \quad t \in [0, \check{\tau}), \quad 1 \le j \le 3N, \tag{1.18}$$

where $\check{b}_j(\mathbf{s})$ *is defined as in Lecture 1 with* Ψ_s *being replaced by* $\check{\Psi}_s$.

Let $\{F_t\}$ be a $\mathrm{SKLE}_{\alpha,b}$. Since $\{F_t\}$ depends also on the initial value $(\xi, \mathbf{s})$ for SDE (1.1)-(1.2),
we shall write $\mathrm{SKLE}_{\alpha,b}$ more precisely as $\mathrm{SKLE}_{\xi,\mathbf{s},\alpha,b}$ occasionally.

Recall that, for an $\mathbb{H}$-hull $A \subset D(\mathbf{s})$, $\tau_A = \inf\{t > 0 : \overline{F}_t \cap \overline{A} \neq \emptyset\}$.

Let $\{\check{F}_t\}_{\{t<\check{\tau}_A\}}$ be the half-plane capacity reparametrization of the image hulls $\{\widetilde{F}_t = \Phi_A(F_t)\}_{\{t<\tau_A\}}$ specified by (1.15).

$\mathrm{SKLE}_{\alpha,b}$ is said to satisfy the locality property
if, for the $\mathrm{SKLE}_{\xi,\mathbf{s},\alpha,b}$ $\{F_t\}$ with an arbitrarily fixed $(\xi, \mathbf{s}) \in \mathbb{R} \times S$ and for any $\mathbb{H}$-hull $A \subset D(\mathbf{s})$,
$\{\check{F}_t,\ t < \check{\tau}_A\}$ has the same distribution as $\mathrm{SKLE}_{\Phi_A(\xi),\Phi_A(\mathbf{s}),\alpha,b}$ restricted to $\{t < \tau_{\Phi_A(A)}\}$.

Here $\mathrm{SKLE}_{\alpha,b}$ and $\mathrm{SKLE}_{\Phi_A(\xi),\Phi_A(\mathbf{s}),\alpha,b}$ can live on two different probability spaces.

In what follows, we assume that

$$\alpha \text{ is a positive constant and } \quad b(\xi, \mathbf{s}) = -b_{\mathrm{BMD}}(\xi, \mathbf{s}).$$

Let $M_t = \int_0^t h_s'(\xi(s))dB_s$. By (1.6), $\langle M \rangle_t = \int_0^t h_s'(\xi(s))^2 ds = \widetilde{a}(t)/2$. Hence $\check{B}_t := M_{\widetilde{a}^{-1}(2t)}$ is a Brownian motion.

The seimi-martingale expression (1.14) of $\widetilde{\xi}(t)$ can be converted into

$$\check{\xi}(t) = \Phi_A(\xi(0)) + \eta(t) - \int_0^t b_{\mathrm{BMD}}(\check{\xi}(s), \check{\mathbf{s}}(s))ds + \alpha \check{B}_t, \ t \le \check{\tau}, \quad (1.19)$$

where

$$\eta(t) = \frac{\alpha^2 - 6}{2} \int_0^t \check{h}_s''(\overset{\circ}{\xi}(s)) \cdot \check{h}_s'(\overset{\circ}{\xi}(s))^{-2} ds,$$

for $\check{h}_s'(z) := h_{\widetilde{a}^{-1}(2s)}'(z)$, $\check{h}_s''(z) := h_{\widetilde{a}^{-1}(2s)}''(z)$ and $\overset{\circ}{\xi}(t) := \xi(\widetilde{a}^{-1}(2t))$. Note that since $h_t(z)$ is univalent in z on the region G_t, $h_t'(z)$ never vanishes there.

Theorem 1.4

$\mathrm{SKLE}_{\alpha,-b_{\mathrm{BMD}}}$ *for a constant* $\alpha > 0$ *enjoys the locality if and only if* $\alpha = \sqrt{6}$.

Proof. "If" part. Assume that $\alpha = \sqrt{6}$.
Then (1.19) is reduced to

$$d\check{\xi}(t) = -b_{\mathrm{BMD}}(\check{\xi}(t), \check{\mathbf{s}}(t))dt + \sqrt{6}\, d\check{B}_t. \tag{1.20}$$

Thus $\{\check{F}_t\}$ is an increasing sequence of $\mathbb{H}$-hulls associated with the unique solution $\check{g}_t$ of the Komatu-Loewner equation (1.17), driven by $(\check{\xi}(t), \check{\mathbf{s}}(t))$, which is the unique solution of (1.20) and (1.18).

Therefore $\{\check{F}_t\}_{\{t<\check{\tau}_A\}}$ is $\mathrm{SKLE}_{\Phi_A(\xi),\Phi_A(\mathbf{s}),\sqrt{6},-b_{\mathrm{BMD}}}$ restricted to $\{t < \tau_{\Phi_A(A)}\}$, yielding the 'if' part of the theorem.

"Only if" part. Assume the locality of $\mathrm{SKLE}_{\alpha,-b_{\mathrm{BMD}}}$. Then $\{(\check{\xi}(t)),\check{\mathbf{s}}(t); t\in(0,\check{\tau}_A)\}$ has the same distribution as the solution $\{(\bar{\xi}(t),\bar{\mathbf{s}}(t)); t\in[0,\bar{\tau}_{\Phi_A(A)})\}$ of the equation

$$\bar{\xi}(t)=\Phi_A(\xi)-\int_0^t b_{\mathrm{BMD}}(\bar{\mathbf{s}}(s)-\widehat{\bar{\xi}}(s))ds+\alpha\bar{B}_t \tag{1.21}$$

for some Brownian motion $\bar{B}_t$ coupled with the equation (1.18) with $(\bar{\xi}(t),\bar{\mathbf{s}}(t))$ in place of $(\check{\xi}(t),\check{s}(t))$.
Then we see from (1.19) that $\check{\xi}(t)$ is, under the Girsanov transform generated by the local martingale $-\alpha^{-1}\eta(t)\,d\check{B}_t$, locally equivalent in law to $\bar{\xi}(t)$. It follows that $\eta(t)=0,\ t<\check{\tau}$, almost surely, and accordingly

$$(\alpha^2-6)\int_0^{\widetilde{a}^{-1}(2t)\wedge\tau} h_s''(\xi(s))ds=0,\quad t>0. \tag{1.22}$$

Dividing (1.22) by $\widetilde{a}^{-1}(2t)$ and then letting $t\downarrow 0$, we get $(\alpha^2-6)\Phi_A''(\xi)=0$ for every $\xi\in\partial\mathbb{H}\setminus\overline{A}$ because we can check that $h_s''(z)$ converges to $\Phi_A''(z)$ locally uniformly as $s\downarrow 0$.
If $\alpha^2\neq 6$, then $\Phi_A''(\xi)=0$ for every $\xi\in\partial\mathbb{H}\setminus A$, This would imply that Φ_A is an identity map, which is impossible unless $A=\emptyset$.

Some overview

1. The chordal SLE_κ can be viewed as a special case of $\mathrm{SKLE}_{\alpha,b}$ for the upper half-plane $\mathbb{H}$ with no slit and for $\alpha = \sqrt{\kappa},\ b = 0$.

In this case, $b_{\mathrm{BMD}} = 0$ and Theorem 1.4 in the above says that SLE_κ enjoys the locality if and only if $\kappa = 6$.

Also Theorem 1.1 reads

$$\frac{d\widetilde{g}_s^0(z)}{ds} = \frac{2|h_s'(\xi(s)|^2}{\widetilde{g}_s^0(z) - h_s(\xi(s))} \tag{2.1}$$

for the canonical Riemann map $\widetilde{g}_s^0(z)$ associated with the image hulls $\{\widetilde{F}_t^0 = \Phi_A(F_t^0)\}$ of SLE_κ $\{F_t^0\}$.

The locality property of SLE_6 was first observed by G. Lawler, O. Schramm and W. Werner.

The present definition of the locality for SKLE is basically taken from

[LSW] G. Lawler, O. Schramm and W. Werner, Conformal restriction: the chordal case, *J. Amer.Math. Soc.* **16** (2003), 917-955

But the present proof may be new even for the SLE as its special case in the following sense.

The generalized Loewner equation (2.1) is well-known to hold in the right time-derivative sense:
See page 96 of

[L] G.F. Lawler, *Conformally Invariant Processes in the Plane*, Mathematical Surveys and Monographs, AMS, 2005

In order to make it a genuine ODE,
we need to verify the continuity of $\widetilde{g}_t(z)$ in t,

which is certainly valid if $\kappa \leq 4$ because $\{F_t\}$ is then generated by a simple curve as has been shown in

[RS] S. Rohde and O. Schramm, Basic properties of SLE, *Ann. Math.* **161** (2005), 879-920

and so is $\{\widetilde{F}_t\}$, yielding the left continuity of $\widetilde{g}_t(z)$ (Section 6 of [CFR]).

The only available way to prove such continuity property for SLE_κ with $\kappa > 4$ seems to be using an analogous but simpler version of the proof of Step I in the above.

However this method does not seem to work for the perturbations of domains by a locally real conformal map formulated in $\S 4.6.1$, $\S 6.3$ of [L] and for an exponential map relating the chordal SLE to the radial SLE in $\S 4.6.3$, $\S 6.5$ of [L].

2. The joint continuity of $h_t(z), h_t'(z), h_t''(z)$ derived from that of $\widetilde{g}_t(z)$ in Step III
is crucial to legitimate frequent usages of a generalized Itô formula presented in Exercise (3.12) of

[RY] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 1999

Incidentally, in this Exercise, 4 conditions i), ii), iii), iv) on an adapted random function $g(x, \omega, u)$ are required for the validity of a generalized Itô formula for its composite with a continuous semi-martingale.
But, additional conditions
ii)' $g_{xx}(x, \omega, u)$ is locally bounded in (x, u) a.s.
iv)' $g_u(x, \omega, u)$ is locally bounded in (x, u) a.s.
should be added to ii) and iv), respectively.
(by private communications with Masanori Hino)

The joint continuity of $g_u(x, u), g_x(x, u).g_{xx}(x, u)$ suffices accordingly.

3. An SKLE is produced by a pair $(\xi(t), \mathbf{s}(t))$ of a motion $\xi(t)$ on $\partial\mathbb{H}$ and a motion $\mathbf{s}(t)$ of slits via Komatu-Loewner equation,
while an SLE is produced by a motion on $\partial\mathbb{H}$ alone via Loewner equation.
They are subject to different mechanisms.
Nevertheless. as a family of random growing hulls,
It can be shown that $\mathrm{SKLE}_{\sqrt{6},-b_{\mathrm{BMD}}}$ after a reparametrization has the same distribution as chordal SLE_6.
Furthermore, it can be shown that when α is constant,
$\mathrm{SKLE}_{\alpha,b}$ up to some random hitting time and modulo a time change has the same distribution as the chordal SLE_{α^2} under suitable Girsanov transform.
In view of [RS], $\mathrm{SKLE}_{\alpha,b}$ $\{F_t\}$ for a constant α is generated by a continuous curve γ in the sense that $\mathbb{H} \setminus F_t$ coincides
with the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$ for each $t > 0$ a.s,
and γ is simple for $0 < \gamma \leq 4$.
See

[CFS] Z.-Q. Chen, M, Fukushima and H. Suzuki, On some relations of stochastic Komatu-Loewner evolutions to SLE, Preprint

4. Perturbations of an SKLE on a multiply connected domain can not be controlled by those of an SLE.

In order to establish the locality of $\mathrm{SKLE}_{\sqrt{6},-b_{\mathrm{BMD}}}$,
we therefore need to work, as in the proof of the 'if' part of Theorem 1.4 with perturbations of standard slit domains.

It can not be obtained as a consequence of the locality of the chordal SLE_6.

By the same reason, the locality of radial SLE_6 does not directly follow from that of chordal SLE_6.

5. 'Only if' part of Theorem 1.4 seems to be new not only for SKLE but for SLE.

6. We say that SLE_κ $\{F_t^0\}$ has the restriction property if

for any $\mathbb{H}$-hull A, conditioned on $\tau_A = \infty$, $\{F_t^0\}$ has the same distribution as its image hulls $\{\Phi_A(F_t^0)\}$.

It has been shown $\mathrm{SLE}_{8/3}$ enjoys the restriction property but does not for $\kappa \neq 8/3,\ 0 < \kappa \leq 4$.

But we can hardly expect a straightforward generalization of the restriction property to $\mathrm{SKLE}_{\sqrt{8/3},-b_{\mathrm{BMD}}}$
due to the aftereffect of the second order BMD-domain constant

$$c_{\mathrm{BMD}}(\xi, D) = 2\pi \lim_{z\to\xi}(\Psi'_D(z,\xi) - 1/[\pi(z-\xi)^2]).$$

7 For $0 < \kappa \leq 4$, SLE_κ is generated by a simple curve
and we may consider conditional processes by specifying its end point.
D. Zhan

[Z] D. Zhan, Restriction properties of annulus SLE, *J. Stat. Phys.* **146(5)**(2012), 1026-1058

discusses the restriction property and the reversibility for annulus SLE_κ specified this way.
G. Lawler

[L2] G. F. Lawler, Defining SLE in multiply connected domains with the Brownian loop measure, arXiv:1108.4364 2011

considers SLE's for more general multiply connected domains using the Brownian loop measure and compares it with Zhan's one in annulus case.

8 Y. LeJan

[Le] Yves LeJan, Markov paths, loops and fields, École d'Été de Probabilités de Saint-Flour, 2008, Lecture Notes in Math. 2026, Springer, 2011

considers loop measures for a general symmetric Markov chain and mention a prospect to extend them to
a general symmetric Markov process with Green density function.