Locality property and a related continuity problem for SLE and SKLE I

Masatoshi Fukushima (Osaka)

joint work with Zhen Qing Chen (Seattle)

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確率解析とその周辺

Osaka University, Σ-hall

Locality property of SLE

$\mathbb{H}$: upper half plane

$A \subset \mathbb{H}$ is called an $\mathbb{H}$-hull if A is a bounded closed subset of $\mathbb{H}$ and $\mathbb{H} \setminus A$ is simply connected

Given an $\mathbb{H}$-hull A, there exists a unique conformal map f (one-to-one analytic function) from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ satisfying a hydrodynamic normalization

$$f(z) = z + \frac{a}{z} + o(1/|z|), \; z \to \infty.$$

Such a map f will be called a canonical Riemann map from $\mathbb{H} \setminus A$ and a is called the half-plane capacity of f.

For $\gamma = \{\gamma(t) : 0 \le t < t_\gamma\}$: a Jordan arc with $\gamma(0) \in \partial\mathbb{H}, \; \gamma(t) \in \mathbb{H}, \; \forall t < t_\gamma$,
let g_t^0 be the canonical Riemann map from $\mathbb{H} \setminus \gamma[0, t]$ with the half-plane capacity a_t.

See Figure 2 (with no slit)

Repametrize γ as $a_t = 2t$.

g_t^0 then satisfies a simple ODE called the chordal Loewner equation

$$\frac{dg_t^0(z)}{dt} = -2\pi\Psi^{\mathbb{H}}(g_t^0(z), \xi(t)) \quad \text{for} \quad \Psi^{\mathbb{H}}(z,\xi) = -\frac{1}{\pi}\frac{1}{z-\xi},\ z \in \mathbb{H},\ \xi \in \partial\mathbb{H}. \tag{1.1}$$

Here $\xi(t) = g_t^0(\gamma(t))$, which is a continuous function taking value in the boundary $\partial\mathbb{H}$.

$\Psi^{\mathbb{H}}(z,\xi)$ may be called the complex Poisson kernel for ABM (absorbing Brownian motion) on $\mathbb{H}$ because

$$\Im\Psi^{\mathbb{H}}(z,\xi) = \frac{1}{\pi}\frac{y}{(x-\xi)^2+y^2}. \quad z = x+iy,$$

Conversely, given a continuous function $\xi(t),\ t \geq 0$, on $\partial\mathbb{H}$, the Cauchy problem of the ODE

$$\frac{dg_t^0(z)}{dt} = -2\pi\Psi^{\mathbb{H}}(g_t^0(z), \xi(t)) \quad g_0^0(z) = z, \quad z \in \mathbb{H}, \tag{1.2}$$

admits a unique solution $\{g_t^0(z),\ t \in [0, t_z^0)\}$
with maximal time interval of existence $[0, t_z^0)$.

If we let

$$F_t^0 = \{z \in \mathbb{H} : t_z^0 \leq t\},$$

then F_t^0 is an $\mathbb{H}$-hull and $g_t^0(z)$ becomes
a canonical Riemann map from $\mathbb{H} \setminus F_t^0$.

The family of growing hulls $\{F_t^0\}$ is called
the Loewner evolution driven by the continuous function $\xi(t)$.

Let $B(t),\ t \geq 0$, be the Brownian motion on $\partial\mathbb{H}$ and κ be a positive constant.
The family of random growing hulls $\{F_t^0\}$ driven by the path of $B(\kappa t),\ t \geq 0$, is called a stochastic Loewner evolution (SLE) and denoted by SLE_κ.

Let $\{F_t^0\}$ be SLE_κ. Consider an $\mathbb{H}$-hull A and the associated canonical Riemann map Φ_A from $\mathbb{H} \setminus A$. Define the image hull by

$$\widetilde{F}_t^0 = \Phi_A(F_t^0), \quad t < \tau,$$

where

$$\tau = \inf\{t > 0 : \overline{F}_t^0 \cap \overline{A} \neq \emptyset\}.$$

We say that SLE_κ $\{F_t^0\}$ has the locality property if the image hulls $\{\widetilde{F}_t^0\}$ has the same distribution as $\{F_t^0\}$ on $t < \tau$ under a certain reparametrization for any $\mathbb{H}$-hull A.

See Figure 3 with no slit and Figure 1 for the relation to the percolation exploration process.

Early in 2000s, G. Lawler, G. Schramm and W. Werner observed that SLE_6 enjoys the locality property.

Basically this can be proved by identifying a time change of the driving process $\widetilde{\xi}(t)$ of the image hulls $\{\widetilde{F}_t^0\}$ with $B(6t)$.

But, in doing so rigorously, we need to verify the joint continuity in $(t, \widetilde{z})$ of the canonical Riemann maps associated with $\{\widetilde{F}_t^0\}$,
that seems to be left unconfirmed
but can be readily shown as will be explaned in the next slide.

In these lectures, we will characterize the locality property of stochatic Komatu-Loewner evolutions for multply connected domains by establishing the stated continuity in this generality using BMD.

See figure 3. The superscript ^{0} indicates no slit.
Since $h_t^0(z)$ is the canonical Riemann map from $\mathbb{H} \setminus g_t^0(A)$,
$\Im(h_t^0(z) - z)$ is a bounded harmonic function on $\mathbb{H} \setminus g_t^0(A)$
and $\Im h_t^0(z) = 0, \quad z \in \partial g_t^0(A)$. Therefore

$$\Im h_t^0(z) = \Im z - \mathbb{E}_z^{\mathbb{H}} \left[\Im Z^{\mathbb{H}}_{\sigma_{g_t^0(A)}} \; ; \; \sigma_{g_t^0(A)} < \infty \right],$$

for the absorbing Brownian motion(ABM) $(Z_t^{\mathbb{H}}, \mathbb{P}_z^{\mathbb{H}})$ on $\mathbb{H}$. Define

$$q_t^0(z) = \Im g_t^0(z) - \mathbb{E}_z^{\mathbb{H}} \left[\Im g_t^0(Z^{\mathbb{H}}_{\sigma_A}); \sigma_A < \infty \right], \quad z \in \mathbb{H} \setminus F_t^0 \setminus A,$$

which is jointly continuous in (t, z)
because $g_t^0(z)$ is a solution of an ODE (1.2).
Due to the invariance of the absorbing BM under the conformal map g_t^0,
we have $\qquad \Im h_t^0(g_t^0(z)) = q_t^0(z)$.
Since $\widetilde{g}_t^0 = h_t^0 \circ g_t^0 \circ \Phi_A^{-1}$, we obtain for each $T \in (0, \tau)$

$$\Im \widetilde{g}_t^0(z) = q_t^0(\Phi_A^{-1}(z)), \quad t \in [0, T], \quad z \in \mathbb{H} \setminus \widetilde{F}_T^0 \setminus \overline{\widetilde{A}},$$

Stochastic Komatu-Loewner evolution (SKLE)

A domain $D \subset \mathbb{H}$ is called a standard slit domain if

$$D = \mathbb{H} \setminus \{C_1, \cdots, C_N\}$$

where $C_k,\ 1 \leq k \leq N$, are mutually disjoint line segments in $\mathbb{H}$ parallel to $\partial\mathbb{H}$.
Denote by $\mathcal{D}$ the collection of all labelled standard slit domains.
Given $D \in \mathcal{D}$ and an $\mathbb{H}$-hull $A \subset D$, there exists a unique conformal map from $D \setminus A$ onto another stadard slit domain
satisfying a hydrodynamic normalization.
We call such a map the canonical conformal map from $D \setminus A$ We fix $D \in \mathcal{D}$ and consider a Jordan arc

$$\gamma : [0, t_\gamma) \to \overline{D}, \quad \gamma(0) \in \partial\mathbb{H},\ \gamma(0, t_\gamma) \subset D,\ 0 < t_\gamma \leq \infty.$$

For each $t \in [0, t_\gamma)$, let $g_t(z)$ be the canonical conformal map from $D \setminus \gamma[0, t]$.
The arc γ can be reparametrized in a way that the half-plane capacity of g_t equals $2t$.

Define

$$\xi(t) = g_t(\gamma(t)) \ (\in \partial\mathbb{H}), \quad 0 \le t < t_\gamma$$

that is continuous in t.

See Figure 2.

Analogously to the chordal Loewner equation (1.1),
the family $g_t(z)$ of conformal maps satisfies
the chordal Komatu-Loewner equation

$$\frac{dg_t(z)}{dt} = -2\pi\Psi_{D_t}(g_t(z), \xi(t)), \quad z \in D \setminus \gamma[0,t]. \tag{2.1}$$

where $\Psi_{D_t}(z,\xi)$ is the BMD-complex Poisson kernel
for the image domain $D_t = g_t(D) \in \mathcal{D}$
that will be explained in the next three slides.

Let Z^0 be the absorbing Brownian motion on D. Let

$$D^* = D \cup K^*, \quad K^* = \{c_1^*, \cdots, c_N^*\}$$

be the space obtained from $\mathbb{H}$ by identifying every point of each slit C_k as a single point c_k^*, $1 \le k \le N$ with the quotient topology.
The Lebesgue measure m on D is extended to D^* by letting $m(K^*) = 0$.

As has been shown in §7.7 of

[CF1] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Changes, and Boundary Theory*, Princeton University Press, 2012

there exists uniquely a diffusion process Z^* on D^* that is an m-symmetric extension of Z^0 with no sojourn nor killing on K^*.

We call Z^* the Brownian motion with darning (BMD) for D.

In the one slit case ($N = 1$), the unique existence of BMD was established by

[FT] M. Fukushima and H. Tanaka, Poisson point processes attached to symmetric diffusions, *Ann.Inst.Henri Poincaré Probab.Statist.* **41**(2005), 419-459

for a general symmetric diffusion and actually BMD was constructed by using Itô's Poisson point process of excursions around the slit,

and it has been identified with Lawler's excursion reflected Brownian motin (ERBM) by

[CF2] Z.-Q. Chen and M. Fukushima, One point reflection, *Stochastic Process Appl.* **125** (2015), 1368-1393

Let $Z^* = \{Z_t^*, \zeta^*, \mathbb{P}_z^*\}$ be the BMD on $D^* = D \cup \{c_1^*, \cdots, c_N^*\}$

A function u on D^* is called BMD-harmonic if u is continuous on D^* and satisfies the usual averaging property that for any relatively compact open set O_1 with $\overline{O}_1 \subset D^*$,

$$\mathbb{E}^*_z\left[|u(Z^*_{\tau_{O_1}})|\right] < \infty \quad \text{and} \quad \mathbb{E}^*_z\left[u(Z^*_{\tau_{O_1}})\right] = u(z) \text{ for every } z \in O_1.$$

Any BMD-harmonic function u is not only harmonic on D in the ordinary sense but also it admits an analytic function f on D with $\Im f = u$ uniquely up to an additive real constant.
For $D \in \mathcal{D}$, let $K^*_D(z,\xi),\ z \in D,\ \xi \in \partial\mathbb{H}$, be the Poisson kernel expressiong any bounded BMD-harmonic function u for D as

$$u(z) = \int_{\partial\mathbb{H}} K^*_D(z,\xi)u(\xi)d\xi, \quad z \in D.$$

Since $K^*_D(z,\xi)$ is BMD-harmonic in z, there exists a unique analytic function $\Psi_D(z,\xi)$ in $z \in D$ with $\Im\Psi_D(z,\xi) = K^*_D(z,\xi)$ under the normalization condition $\lim_{z\to\infty}\Psi_D(z,\xi) = 0$.

$\Psi_D(z,\xi)$ is called the BMD-complex Poisson kernel for $D \in \mathcal{D}$.
$\Psi_D(z,\xi)$ admits an explicit expression in terms of the Green function of D.

We return to the chordal Komatu-Loewner equation (2.1) induced by the Jordan arc γ.
$\gamma(t)$ induces not only the motion of $\xi(t) = g_t(\gamma(t)) \in \partial\mathbb{H}$
but also the motion of the image domain $D_t = f_t(D)$.

Denote by $z_j(t) = x_j(t) + iy_j(t)$, $z_j'(t) = x_j'(t) + iy_j(t)$, the left and right endpoints of the j-th slit $C_j(t)$ of the image domain D_t. Then we can derive from the Komatu-Loewner differential equation (2.1)

$$\begin{cases} \frac{d}{dt} y_j(t) = -2\pi \Im \Psi_{D_t}(z_j(t), \xi(t)), \\ \frac{d}{dt} x_j(t) = -2\pi \Re \Psi_{D_t}(z_j(t), \xi(t)), \\ \frac{d}{dt} x_j'(t) = -2\pi \Re \Psi_{D_t}(z_j'(t), \xi(t)), \quad 1 \le j \le N. \end{cases} \tag{2.2}$$

which is called the K-L slit equation.

(2.2) particularly means that the motion of D_t is determined by the motion of $\xi(t)$.

For $D, \widetilde{D} \in \mathcal{D}$, define their distance $d(D, \widetilde{D})$ by

$$d(D, \widetilde{D}) = \max_{1 \le i \le N} (|z_i - \widetilde{z}_i| + |z_i' - \widetilde{z}_i'|),$$

where, for $D = \mathbb{H} \setminus \{C_1, C_2, \cdots, C_N\}$,
It is convenient to introduce an open subset S of the Euclidean space $\mathbb{R}^{3N}$ by

$$S = \{(\mathbf{y}, \mathbf{x}, \mathbf{x}') \in \mathbb{R}^{3N} : \mathbf{y} > \mathbf{0},\ \mathbf{x} < \mathbf{x}', \\ \text{either } x_j' < x_k \text{ or } x_k' < x_j \text{ whenever } y_j = y_k,\ j \neq k\}.$$

By the correspondence

$$z_k = x_k + iy_k,\ z_k' = x_k' + iy_k,\ 1 \le k \le N,$$

the space $\mathcal{D}$ can be identified with S as a topological space.
The point in S (resp. $\mathcal{D}$) corresponding to $D \in \mathcal{D}$ (resp. $\mathbf{s} \in S$) will be denoted by $\mathbf{s}(D)$ (resp. $D(\mathbf{s})$).
$\{D_t : 0 \le t < t_\gamma\}$ is a one parameter subfamily of $\mathcal{D}$.
$\mathbf{s}(D_t)$ is designated by $\mathbf{s}(t)$.
The K-L equation (2.1) can be rewritten as

$$\frac{dg_t(z)}{dt} = -2\pi \Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)), \quad g_0(z) = z \in D. \tag{2.3}$$

Due to the invariance of BMD under the shift map in x-direction,
The K-L slit equation (2.2) can be rewritten as

$$\mathbf{s}_j(t) - \mathbf{s}_j(0) = \int_0^t b_j(\mathbf{s}(s) - \widehat{\xi}(s))ds, \quad t \geq 0, \quad 1 \leq j \leq 3N, \qquad (2.4)$$

where

$$b_j(\mathbf{s}) = \begin{cases} -2\pi \Im \Psi_{\mathbf{s}}(z_j, 0), & 1 \leq j \leq N, \\ -2\pi \Re \Psi_{\mathbf{s}}(z_j, 0), & N+1 \leq j \leq 2N, \quad \mathbf{s} \in S. \\ -2\pi \Re \Psi_{\mathbf{s}}(z_j', 0), & 2N+1 \leq j \leq 3N. \end{cases} \qquad (2.5)$$

and $\widehat{\xi}$ denotes the $3N$-vector with the first N-componets equal to 0 and the next $2N$-components equal to ξ.
Let us consider the next locall Lipschitz condition for a real function $f = f(\mathbf{s})$ on S.

(L) For any $\mathbf{s}^{(0)} \in S$ and any bounded open interval $J \subset \mathbb{R}$,
there exist a neigborhood $U(\mathbf{s}^{(0)}) \subset S$ of $\mathbf{s}^{(0)}$ and a constant $L > 0$ with

$$|f(\mathbf{s}^{(1)} - \widehat{\xi}) - f(\mathbf{s}^{(2)} - \widehat{\xi})| \leq L\, |\mathbf{s}^{(1)} - \mathbf{s}^{(2)}| \quad \forall \mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in U(\mathbf{s}^{(0)}) \quad \forall \xi \in J.$$

Proposition 2.1

(i) $b_j(\mathbf{s}),\ 1 \le j \le 3N$, *satisfies the condition* **(L)**.
(ii) *Given any real continuous function* $\xi(t)$ *on* $[0, \infty)$ *and any* $\mathbf{s}^{(0)} \in S$. *there exists a unique solution* $\mathbf{s}(t)$ *of the K-L slit equation* (2.4) *satisfying* $\mathbf{s}(0) = \mathbf{s}^{(0)}$.

(i) is a consequence of the Lipschitz continuity of the BMD complex Poisson kernel Ψ_D shown in

[CFR] Z.-Q.Chen, M. Fukushima and S. Rohde, Chordal Komatu-Loewner equation and Brownian motion with darning in multiply connected domains, to appear in *Trans. Amer. Math. Soc.*

(ii) follows from (i).

For a given real continuous function $\xi(t),\ t \in [0,\infty)$,
let $\mathbf{s}(t),\ t \in [0,\zeta)$ be the unique solution of (2.4)
with the maximal interval $[0,\zeta)$ of existence.

Writing $D_t = D(\mathbf{s}(t)) \in \mathcal{D},\ t \in [0,\zeta)$, we set

$$G = \bigcup_{t\in[0,\zeta)} \{t\} \times D_t, \tag{2.6}$$

which is a subdomain of $[0,\zeta) \times \mathbb{H}$ as $t \mapsto D_t$ is continuous.

We then substitute $(\xi(t), \mathbf{s}(t))$ into the Komatu-Loewner equation (2.1)
and consider the Cauchy problem

$$\frac{d}{dt}z(t) = -2\pi\Psi_{\mathbf{s}(t)}(z(t),\xi(t)), \quad z(0) = z \in D(= D_0 = D(\mathbf{s}(0))). \tag{2.7}$$

Theorem 2.2

(i) *For each* $z \in D := D(\mathbf{s}(0))$, *the equation* (2.7) *admits a unique solution* $g_t(z)$, $t \in [0, t_z)$, *passing through* G. *Here* $[0, t_z)$, $t_z > 0$, *is its maximal interval existence. Further*

$$\lim_{t \uparrow t_z} \Im g_t(z) = 0, \quad \text{If } t_z < \zeta \text{ then } \lim_{t \uparrow t_z} |g_t(z) - \xi(t_z)| = 0.$$

(ii) *Let* $F_t = \{z \in D : t_z \le t\}$, $t > 0$ F_t *is an* $\mathbb{H}$-*hull..*
(iii) g_t *is a canonical conformal map from* $D \setminus F_t$ *with the half-plane capacity* $2t$.
(iv) *The increasing family* $\{F_t\}$ *of* $\mathbb{H}$-*hulls is right continuous with limit* $\xi(t)$ *in the following sense:*

$$\bigcap_{\delta > 0} \overline{g_t(F_{t+\delta} \setminus F_t)} = \{\xi(t)\} \quad t \in [0, \zeta). \tag{2.8}$$

The increasing family of the growing $\mathbb{H}$-hulls $\{F_t\}$ in Theorem 2.2 is called the Komatu-Loewner evolution driven by a continuous function $\xi(t)$.

By making an analogous consideration to Schramm, we can verify that, as a driving function $\xi(t)$, the following specific choice of a random process is natural:
A function $f(\mathbf{s})$ on the space S is called homogeneous with degree 0 (resp. -1) if $f(c\mathbf{s}) = f(\mathbf{s})$ (resp. $f(c\mathbf{s}) = c^{-1}f(\mathbf{s})$) for every $c > 0$.
For given real homogeneous functions $\alpha,\ b$ on S with degree $0,\ -1$, respectively, both satisfying the Lipschitz condition **(L)**,
we consider the stochastic differential equation

$$\xi(t) = \xi + \int_0^t \alpha(\mathbf{s}(s) - \widehat{\xi}(s))dB_s + \int_0^t b(\mathbf{s}(s) - \widehat{\xi}(s))ds, \qquad (2.9)$$

(B_s is a standard Brownian motion),
coupled with the K-L slit equation (2.4)

$$\mathbf{s}_j(t) - \mathbf{s}_j(0) = \int_0^t b_j(\mathbf{s}(s) - \widehat{\xi}(s))ds, \quad t \geq 0, \quad 1 \leq j \leq 3N.$$

It is known that $b_j(\mathbf{s})$ defined by (2.5) is homogeneous with degree -1 and satisfies the condition **(L)** by [CFR].

Let $(\xi(t), \mathbf{s}(t))$ be the strong solution of the system of the equations (2.9) and (2.4).

The K-L evolution driven by $\xi(t)$ will be called the stochastic K-L evolution driven by the solution of the SDE (2.9), (2.4) and denoted by $\mathrm{SKLE}_{\alpha,b}$.

The results of this section are taken from

[CF3] Z.-Q. Chen and M. Fukushima, Stochastic Komatu-Loewner evolution and BMD domain constant, arXiv:1410.8257vl