Locality property and a related continuity problem for SLE and SKLE

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The *Schramm Loewner evolution* SLE_κ for $\kappa > 0$ is a family of random growing hulls $\{F_t\}$ in the upper half-plane $\mathbb{H}$ driven by $\xi(t) = B(\kappa t)$ through the *Loewner differential equation*, where $B(t),\ t \geq 0$, is the standard Brownian motion on the boundary $\partial\mathbb{H}$. Early in the 2000s, G. Lawler, G. Schramm and W. Werner observed that SLE_6 enjoys the *locality property* in the sense that, if $\mathbb{H}$ is perturbed away from the hulls, the family of perturbed hulls $\{\widetilde{F}_t\}$ under a due reparametrization has the same distribution as the unperturbed one $\{F_t\}$ (see Figure 1 for percolation). This property can be shown by comparing the driving process $\widetilde{\xi}(t)$ of $\{\widetilde{F}_t\}$ with $B(6t)$ in principle.

But, in doing so rigorously, one need to verify the joint continuity in $(t, \widetilde{z})$ for the family of conformal maps $\widetilde{g}_t(\widetilde{z})$ associated with $\{\widetilde{F}_t\}$, that seems to be left unconfirmed. In this lectures, we will characterize the locality property of stochastic Komatu-Loewner evolutions for multiply connected domains by establishing the stated continuity in this generality.

A *standard slit domain* is a domain of the type $D = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j$ for mutually disjoint line segments $C_j \subset \mathbb{H}$ parallel to $\partial\mathbb{H}$. Consider a set $S \subset \mathbb{R}^{3N}$ representing the collection of all labelled slits. Let $(\xi(t), \mathbf{s}(t)) \in \partial\mathbb{H} \times S$ be the strong solution of an SDE such that a diffusion coefficient α and a drift coefficient b of $\xi(t)$ are homogeneous functions of degree 0 and -1, respectively, both satisfying a local Lipschitz continuity condition, while each component of $\mathbf{s}(t)$ has only a drift coefficient determined by the trace to the slits of the *BMD complex Poisson kernel* that is known to be locally Lipschitz continuous.

A *stochastic Komatu-Loewner evolution* denoted by $\mathrm{SKLE}_{\alpha,b}$ is a family of random growing hulls $\{F_t\}$ in a standard slit domain D driven by $(\xi(t), \mathbf{s}(t))$ through the *Komatu-Loewner differential equation*. Let b_{BMD} be the *BMD-domain constant* that describes the discrepancy of a standard slit domain from $\mathbb{H}$ relative to BMD (*Brownian motion with darning*).

Theorem 0.1 $\mathrm{SKLE}_{\alpha,-b_{\mathrm{BMD}}}$ *for a positive constant* α *enjoys the locality property if and only if* $\alpha = \sqrt{6}$.

We use a probabilistic expression of $\Im h_t(z)$ in terms of the BMD and the absorbing Brownian motion (ABM) on D_t and combine it with the conformal invariance of BMD and ABM to obtain an expression of $\Im \widetilde{g}_t(\widetilde{z})$ in terms of $g_t(z)$ (see Figure 3). Note that $g_t(z)$ is jointly continuous as the solution of ODE (KL-equation). This is the way to prove the desired joint continuity of $\widetilde{g}_t(\widetilde{z})$, which additionally yields the joint continuity of $h_t(z), h_t'(z), h_t''(z)$. The last property is crucial to legitimate a use of a generalized Itô formula on a composite of a semi-martingale and a random smooth function formulated by Revuz-Yor in getting an explicit semi-martingale expression of the driving process $\widetilde{\xi}(t)$ of the image hulls $\{\widetilde{F}_t\}$.

Some open problems related to SKLE will be also discussed.