Phase transitions in a control problem

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A control problem

We consider the following controlled system of SDE

$$\begin{aligned} dX_t &= \gamma dW_t + d\Lambda_t, \\ dY_t &= X_t dW_t + d\|\Lambda\|_t - \beta dt, \end{aligned}$$

where $\beta > 0$ and $\gamma \neq 0$ are constants, W is a standard Brownian motion and $\|\Lambda\|$ is the total variation of our control Λ that we require to be an adapted process of finite variation.

The problem, motivated by a financial practice of hedging under transaction costs, is to minimize

$$\limsup_{T\to\infty} \frac{1}{T} E[Y_T^2] = \limsup_{T\to\infty} \frac{1}{T} E\Big[\int_0^T (X_t^2 - 2\beta Y_t) dt + 2\int_0^T Y_t d\|\Lambda\|_t\Big].$$

1 dim ? 2 dim ?

Example: Bang-bang control

An example of control:

$$\mathrm{d}\Lambda_t = -\alpha \operatorname{sgn}(X_t)\mathrm{d}t$$

with $\alpha > 0$. Then

$$\mathrm{d}X_t = \gamma \mathrm{d}W_t - \alpha \operatorname{sgn}(X_t)\mathrm{d}t,$$
$$\mathrm{d}Y_t = X_t \mathrm{d}W_t + (\alpha - \beta)\mathrm{d}t.$$

In this case,

$$\frac{1}{T}E[Y_T^2] = \frac{1}{T}E[\int_0^T X_t^2 \mathrm{d}t] + 2(\alpha - \beta)E[\int_0^T X_t \mathrm{d}t] + (\alpha - \beta)^2 T \to \infty$$

unless $\alpha = \beta$. The erogodic disribution of X is Laplace (two-sided exponetial).

Example: Singular control

Another example of control:

$$\mathrm{d}\Lambda_t = \mathrm{d}L_t - \mathrm{d}R_t,$$

where L and R are nondecreasing processes such that

$$L_t = \int_0^t 1_{\{X_t=-b\}}\mathrm{d}L_t, \quad R_t = \int_0^t 1_{\{X_t=b\}}\mathrm{d}R_t$$

and $|X_t| \le b$, $b > 0$. Such a control exists, as the solution of the Skorokhod equation

$$\mathrm{d}X_t = \gamma\mathrm{d}W_t + \mathrm{d}L_t - \mathrm{d}R_t.$$

Then, the erogodic distribution of X is $U[-b, b]$, and

$$Y_T = \int_0^T X_t\mathrm{d}W_t + L_T + R_T - \beta T.$$

A class of controls

In this talk, we focus on a restricted class of control

$$d\Lambda = -\mathrm{sgn}(X_t)\gamma^2 c(X_t)\mathrm{d}t + \mathrm{d}L_t - \mathrm{d}R_t, \tag{1}$$

where c is a nonnegative continuous even function on an interval $[-b, b]$, $b > 0$ and L and R are nondecreasing processes with

$$\mathrm{d}L_t = 1_{\{X_t=-b\}}\mathrm{d}L_t, \quad \mathrm{d}R_t = 1_{\{X_t=b\}}\mathrm{d}R_t$$

which keep X stay in $[-b, b]$. Now our control is (b, c).

The idea of the control is to push X towards 0:
1) The abs. conti. part of Λ determined by c pushes X regularly.
2) The other parts are active only when X reaches the boundary of $[-b, b]$ and push X to prevent it from going out of the interval.

Well-defined ?

Such a control exists; in fact, there exists a pathwise unique strong solution (X, L, R) of a Skorokhod SDE

$$\mathrm{d}X_t = \gamma \mathrm{d}W_t - \mathrm{sgn}(X_t)\gamma^2 c(X_t)\mathrm{dt} + \mathrm{d}L_t - \mathrm{d}R_t$$

on $[-b, b]$ when $x \mapsto -\mathrm{sgn}(x)c(x)$ is one-sided Lipschitz. The control Λ is then well-defined by (1).

The optimal control in this restricted class is probably suboptimal for the original problem; however it has a certain advantage in its easy implementation. Also this type of control strategies has appeared in a related context of optimal hedging.

Now, no more sure about the validity of the dynamic programming principle due to that the control Λ refers only to the spot value of X. A (formal) HJB type equation is far from standard forms.

Remind

The system is

$$\mathrm{d}X_t = \gamma \mathrm{d}W_t - \mathrm{sgn}(X_t)\gamma^2 c(X_t)\mathrm{d}t + \mathrm{d}L_t - \mathrm{d}R_t,$$
$$\mathrm{d}Y_t = X_t \mathrm{d}W_t + \gamma^2 c(X_t)\mathrm{d}t + \mathrm{d}L_t + \mathrm{d}R_t - \beta \mathrm{d}t,$$

where L and R are nondecreasing processes with

$$\mathrm{d}L_t = 1_{\{X_t=-b\}}\mathrm{d}L_t, \quad \mathrm{d}R_t = 1_{\{X_t=b\}}\mathrm{d}R_t$$

which keep X stay in $[-b, b]$.

The goal is to find (b, c) which minimizes

$$\limsup_{T\to\infty} \frac{1}{T} E[Y_T^2],$$

where $b > 0$, $c : [-b, b] \to [0, \infty)$; an even continuous function.

A probabilistic approach

Theorem: 1) As $T \to \infty$,

$$\frac{1}{\sqrt{T}}(Y_T - \delta^{b,c}T) \to \mathcal{N}(0, Q^{b,c})$$

in law, where

$$\delta^{b,c} = \frac{\gamma^2}{a} - \beta, \quad a = 2\int_0^b g(x)\mathrm{d}x, \quad g(x) = \exp\left\{-2\int_0^x c(y)\mathrm{d}y\right\}$$

and

$$Q^{b,c} = \frac{2}{a}\int_0^b (x - \gamma h(x))^2 g(x)\mathrm{d}x,$$
$$h(x) = \frac{2}{g(x)}\int_0^x \left(c(y) - \frac{1}{a}\right) g(y)\mathrm{d}y.$$

2)

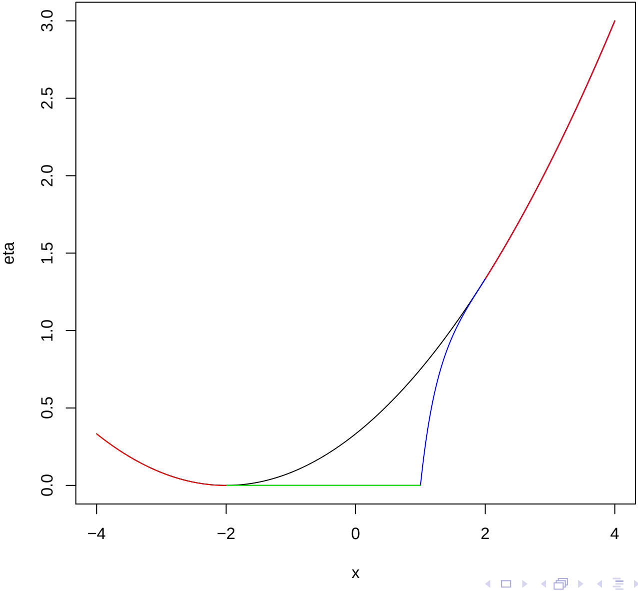
$$\lim_{T\to\infty} \frac{1}{T} E[(Y_T - \delta^{b,c} T)^2] = Q^{b,c}.$$

3)

$$\inf_{\delta^{b,c}=0} Q^{b,c} = \gamma^2 \eta\left(\frac{\gamma}{\beta}\right),$$

where

$$\eta(x) = \begin{cases} 0 & \text{if } -2 < x \le 1, \\ \frac{4}{3}\frac{(x+2)^2(x-1)}{x^3(4-x)} & \text{if } 1 < x < 2, \\ \frac{1}{12}(x+2)^2 & \text{if } |x| \ge 2. \end{cases}$$



eta
0.0
0.5
1.0
1.5
2.0
2.5
3.0
−4
−2
0
2
4
x

Proof of the convergence is...

standard and not difficult.

Denoting by $\mathcal{L}$ the generator of X, we have

$$\frac{1}{\gamma^2}\mathcal{L}H = c - \frac{1}{a}, \quad H'(-b) = -H'(b) = 1,$$

where $H(x) = \int_0^{|x|} h(y)\mathrm{d}y$. Then,

$$\begin{aligned} Y_t = & \int_0^t X_s \mathrm{d}W_s + \gamma^2 \int_0^t c(X_s)\mathrm{d}s + L_t + R_t - \beta t \\ = & H(X_t) - H(X_0) + \int_0^t (X_s - \gamma H'(X_s))\mathrm{d}W_s + \left(\frac{\gamma^2}{a} - \beta\right) t \end{aligned}$$

and so,

$$\frac{1}{\sqrt{T}}(Y_T - \delta^{b,c}T) \approx \frac{1}{\sqrt{T}} \int_0^T (X_s - \gamma H'(X_s))\mathrm{d}W_s.$$

Martingale CLT: If M^n is a sequence of continuous semimartingale with $\langle M^n \rangle_{T_n} \to Q$, then $M^n_{T_n} \to \mathcal{N}(0, Q)$.

Therefore, it suffices to show

$$F_T := \frac{1}{T}\int_0^T f(X_s)\mathrm{d}s - \frac{1}{a}\int_{-b}^{b} f(x)g(|x|)\mathrm{d}x \to 0$$

for all integrable f, which is also easy to see: let

$$\Psi(w) = \frac{1}{\gamma^2}\int_0^w \frac{2}{g(|z|)}\int_{-b}^{z}\left(f(y) - \frac{1}{a}\int_{-b}^{b} f(x)g(|x|)\mathrm{d}x\right)g(|y|)\mathrm{d}y\mathrm{d}z.$$

Then,

$$\mathcal{L}\Psi = f - \frac{1}{a}\int_{-b}^{b} f(x)g(|x|)\mathrm{d}x, \quad \Psi'(-b) = \Psi(b) = 0,$$

and so, $F_T = \left\{\Psi(X_T) - \Psi(X_0) - \gamma\int_0^T \Psi'(X_s)\mathrm{d}W_s\right\}/T \to 0.$

Minimization

The mathematically challenging part is the minimization of $Q^{b,c}$:

$$Q^{b,c} = \frac{2}{a}\int_0^b (x - \gamma h(x))^2 g(x)\mathrm{d}x \to \min$$

under $\delta^{b,c} = 0$, where (recall)

$$h(x) = \frac{2}{g(x)}\int_0^x \left(c(y) - \frac{1}{a}\right) g(y)\mathrm{d}y,$$
$$\delta^{b,c} = \frac{\gamma^2}{a} - \beta, \quad a = 2\int_0^b g(x)\mathrm{d}x$$

and g is a speed measure density:

$$g(x) = \exp\left\{-2\int_0^x c(y)\mathrm{d}y\right\}.$$

Lemma: For each (b, c), there corresponds a unique increasing convex C^2 function y on $[0, 1]$ with $y(0) = 0, y'(0) = a/2$ such that

$$Q^{b,c} = \eta_a[y] := \int_0^1 \left(y(u) + \gamma + \frac{2\gamma}{a}(u-1)y'(u)\right)^2 \mathrm{d}u.$$

Proof: We can show that for all $x \in (0, b)$,

$$h(x) > -1, \quad h'(x) \geq -\frac{2}{a}, \quad c(x) = \frac{h'(x) + 2/a}{2(1 + h(x))}$$

and $h(b) = -1$. Therefore,

$$\begin{aligned} Q^{b,c} =& \frac{2}{a}\int_0^b \frac{(x - \gamma h(x))^2}{1 + h(x)} \exp\left\{-\frac{2}{a}\int_0^x \frac{\mathrm{d}y}{1 + h(y)}\right\} \mathrm{d}x \\ =& \frac{2}{a}\int_0^\infty (x(t) + \gamma(1 - x'(t)))^2 e^{-2t/a} \mathrm{d}t, \end{aligned}$$

where $t = \int_0^{x(t)} \frac{\mathrm{d}y}{1+h(y)}$. Put $y(u) = x(t)$ with $t = -\frac{a}{2}\log(1 - u)$.

Key Lemma

Lemma: Let $\mathcal{Y}_a$ be the set of the increasing convex functions y on $[0,1]$ with $y(0)=0$ and $y'(0)=a/2$. Then,

$$\inf_{y\in\mathcal{Y}_a} \eta_a[y] = \gamma^2 \eta\left(\frac{a}{\gamma}\right),$$

where (recall)

$$\eta_a[y] = \int_0^1 \left(y(u) + \gamma + \frac{2\gamma}{a}(u-1)y'(u)\right)^2 \mathrm{d}u,$$

$$\eta(x) = \begin{cases} 0 & \text{if } -2 < x \le 1, \\ \frac{4}{3}\frac{(x+2)^2(x-1)}{x^3(4-x)} & \text{if } 1 < x < 2, \\ \frac{1}{12}(x+2)^2 & \text{if } |x| \ge 2. \end{cases}$$

Proof for the case $-2 < a/\gamma < 1$: we shall prove $\inf \eta_a[y] = 0$. The key observation here is that

$$y_0(u) = \gamma(1-u)^{-a/2\gamma} - \gamma$$

solves

$$y_0(u) + \gamma + \frac{2\gamma}{a}(u-1)y_0'(u) = 0, \quad y_0(0) = 0, \quad y_0'(0) = \frac{a}{2}$$

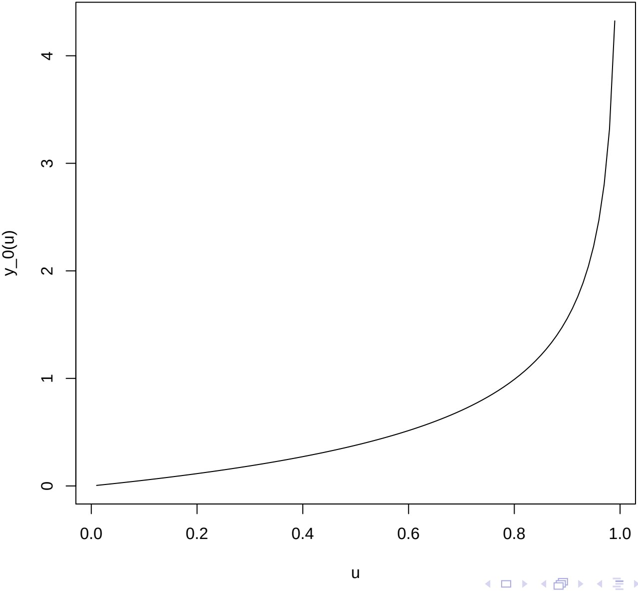
and so, $\eta_a[y_0] = 0$. Note also that y_0 is convex iff $a/\gamma \geq -2$. The only problem is that $y_0(1) = \infty$ when $a/\gamma > 0$ and so, $y_0 \notin \mathcal{Y}_a$.

A solution is to chop $y_0(u)$ at $u = 1 - \epsilon$ and linearly extrapolate it. Then

$$\eta_a[y_\epsilon] = q(1-\epsilon, y_0(1-\epsilon), y_0'(1-\epsilon), 2\gamma/a) = O(\epsilon^{1-a/\gamma}),$$

where

$$q(v, w, z, \theta) = \int_v^1 (w + z(u-v) + \gamma + \theta(u-1)z)^2 \mathrm{d}u.$$



y_0(u)
0
1
2
3
4
0.0
0.2
0.4
0.6
0.8
1.0
u

An important idea for the case $1 < a/\gamma < 2$: For $y \in \mathcal{Y}_a$, let

$$\varphi(u) = (1-u)^{a/2\gamma}(y(u)+\gamma).$$

Then, $\varphi(0) = \gamma$, $\varphi(1) = 0$ and

$$y(u) + \gamma + \frac{2\gamma}{a}(u-1)y'(u) = -\frac{2\gamma}{a}(1-u)^{1-a/2\gamma}\varphi'(u).$$

By the Cauchy-Schwarz, for $u_0 \in [0,1]$,

$$\varphi(u_0)^2 = \left|\int_{u_0}^1 \varphi'(u)\mathrm{d}u\right|^2 \le \int_{u_0}^1 (1-u)^{2-a/\gamma}\varphi'(u)^2\mathrm{d}u \int_{u_0}^1 (1-u)^{a/\gamma-2}\mathrm{d}u.$$

Therefore,

$$\int_{u_0}^1 \left(y(u) + \gamma + \frac{2\gamma}{a}(u-1)y'(u)\right)^2 \mathrm{d}u \ge \frac{4\gamma^2}{a^2}\left(\frac{a}{\gamma}-1\right)(1-u_0)^{1-a/\gamma}\varphi(u_0)^2.$$

In particular, $\inf \eta_a[y] > 0$.

Proof for the case $a/\gamma \geq 2$: Let $\hat{y}(u) = au/2$.

Then, $\hat{y} \in \mathcal{Y}_a$ and $\eta_a[y] = \gamma^2 \eta(a/\gamma)$.

Suppose there exists $y \in \mathcal{Y}_a$ such that $\eta_a[y] < \eta_a[\hat{y}]$. Since a convex function is approximated by piecewise linear convex functions arbitrarily close, we can and do assume y itself is piecewise linear without loss of generality.

Let $u_0 \in (0, 1)$ be the last point where y' jumps. Denote $y'_- = \lim_{u \uparrow u_0} y'(u)$ and $y'_+ = \lim_{u \downarrow u_0} y'(u)$. For $z \geq y'_-$, define $y(\cdot, z)$ as

$$y(u, z) = \begin{cases} y(u) & \text{if } 0 \leq u \leq u_0 \\ y(u_0) + z(u - u_0) & \text{if } u_0 < u \leq 1 \end{cases}$$

(we define a family of piecewise linear convex functions)

Note that $y(\cdot, y'_+) = y$.

$$\int_{u_0}^{1} \left(y(u,z) + \gamma + \frac{2\gamma}{a}(u-1)y'(y,z) \right)^2 \mathrm{d}u = q(u_0, y(u_0), z, 2\gamma/a)$$

and q is quadratic in z. Therefore easy to minimize in z. It turns out that it is minimized by $z = y'_-$. In particular

$$\eta_a[y(\cdot, y'_-)] < \eta_a[y(\cdot, y'_+)] = \eta_a[y].$$

This means that by removing the last jump we get a smaller value.

We can repeat the same argument with $y(\cdot, y'_-)$ instead of y to get an even smaller value. Continue, and eventually all jumps are removed. The final product coincides with $\hat{y}$, which contradicts how y was chosen.

Optimal strategy

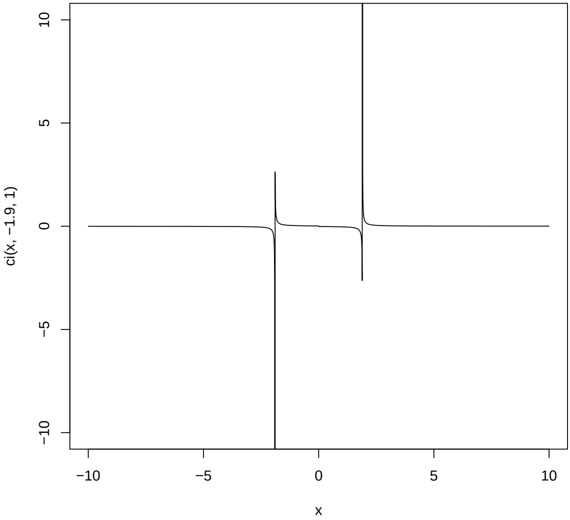
We can give an explicit sequence of controls (b_n, c_n) with $\delta^{b_n,c_n} = 0$ such that Q^{b_n,c_n} converges to the infimum.

In fact, $b_n = \gamma^2/2\beta$ and $c_n = 0$ when $|\gamma| \geq 2\beta$.

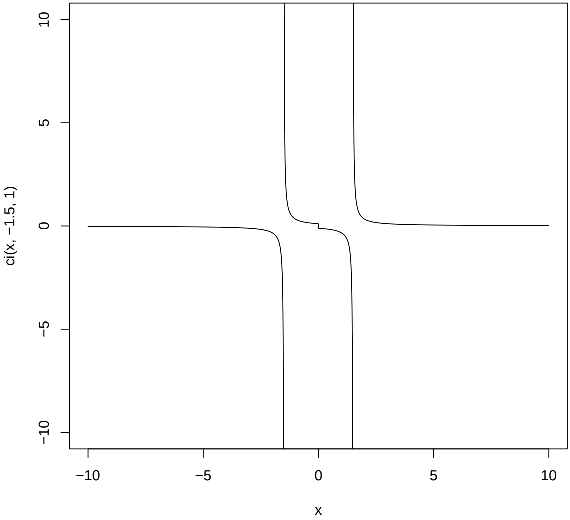
When $|\gamma| < 2\beta$ on the other hand, $b_n \to \infty$ as $n \to \infty$ and the pointwise limit of $c_n(x)$ is given by

$$c_\infty(x) = \frac{\gamma + 2\beta}{2(\gamma - 2\beta l)} \frac{1}{\gamma + |x|} 1_{\{|x| \geq l\}}, \quad l = \frac{2(\gamma - \beta)_+}{4\beta - \gamma}.$$

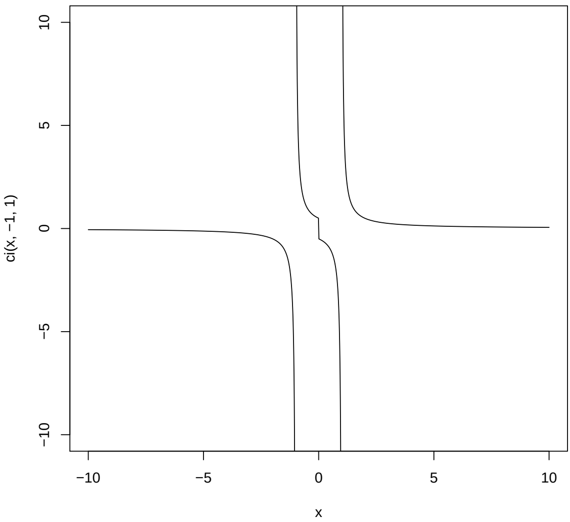
$-\mathrm{sgn}(x)c_\infty(x) : \gamma = -1.9, \beta = 1$



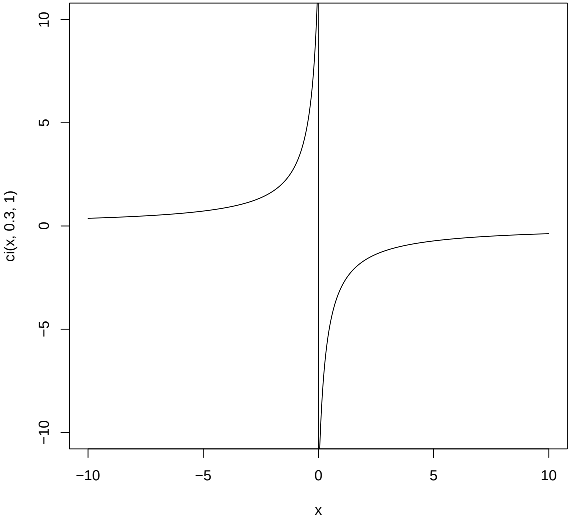
$-\mathrm{sgn}(x)c_\infty(x) : \gamma = -1.5, \beta = 1$



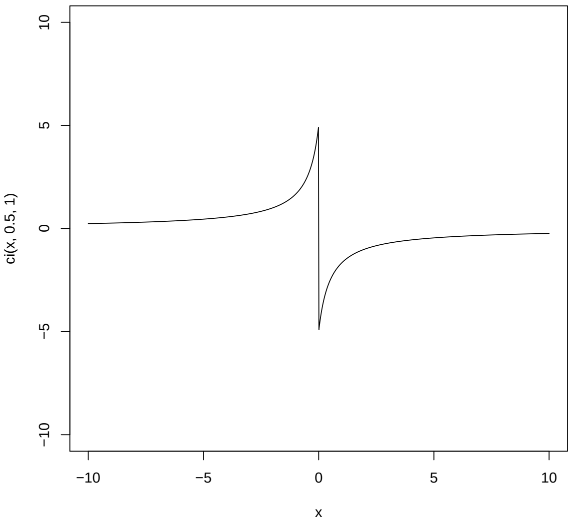
$-\mathrm{sgn}(x)c_\infty(x) : \gamma = -1, \beta = 1$



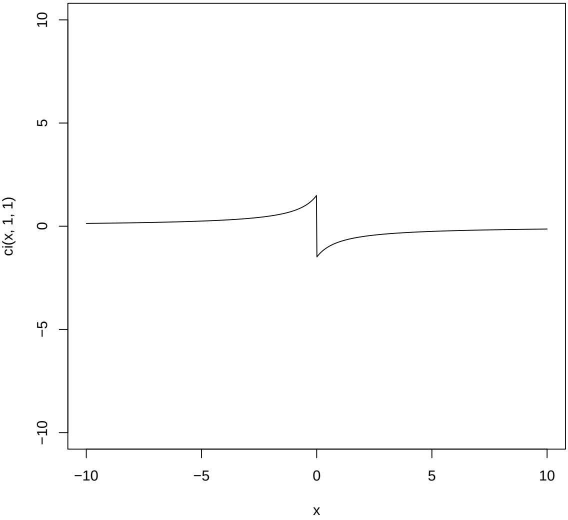
$-\mathrm{sgn}(x) c_\infty(x) : \gamma = 0.3, \beta = 1$



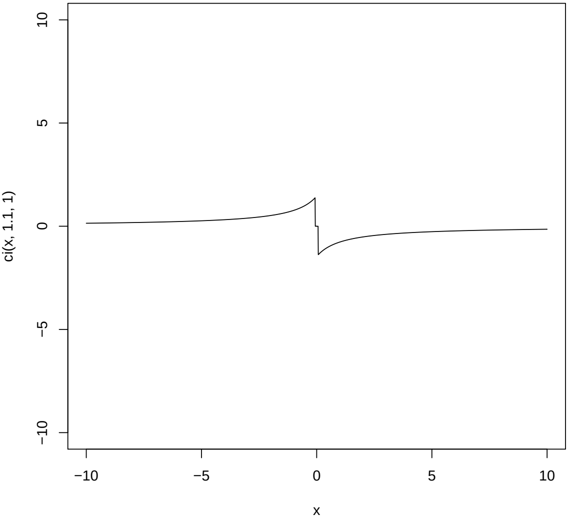
$-\mathrm{sgn}(x)c_\infty(x) : \gamma = 0.5, \beta = 1$



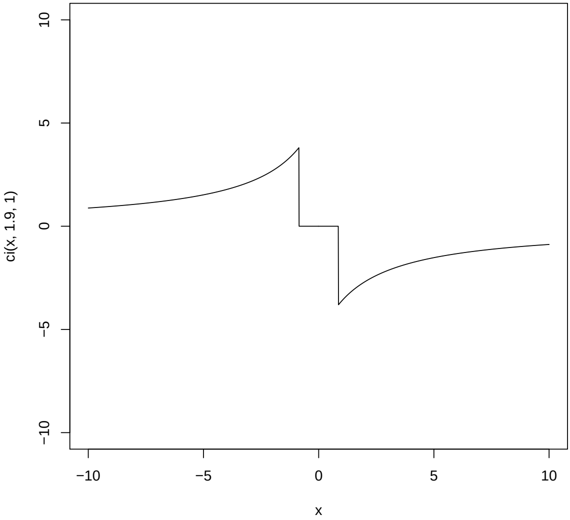
$-\mathrm{sgn}(x)c_\infty(x) : \gamma = 1, \beta = 1$



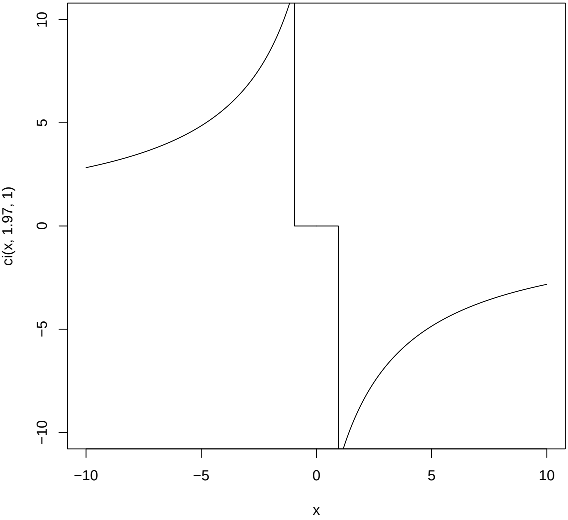
$-\mathrm{sgn}(x)c_\infty(x) : \gamma = 1.1, \beta = 1$



$-\mathrm{sgn}(x)c_\infty(x) : \gamma = 1.9, \beta = 1$



$-\operatorname{sgn}(x)c_\infty(x) : \gamma = 1.97, \beta = 1$



Phase transitions

$x = a/\gamma$	min. limit var.	limit optimal strategy
$x \leq -2$	$(a + 2\gamma)^2/12$	singular control
$-2 < x < 0$	0	regular control with natural boundary
$0 < x < 1$	0	regular control with no boundary
$1 \leq x < 2$	$*$	regular control with no boundary
$x \geq 2$	$(a + 2\gamma)^2/12$	singular control

The phase transition at $a/\gamma = 1$ results from the break of an integrability property.

Ichihara (2015) for phase transitions in controlled Markov chains.

See Cai and Fukasawa (to appear in F&S) for more details.