Phase transitions in a control problem

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We consider the following controlled system of SDE

$$\begin{aligned} \mathrm{d}X_t &= \gamma \mathrm{d}W_t + \mathrm{d}\Lambda_t, \\ \mathrm{d}Y_t &= X_t \mathrm{d}W_t + \mathrm{d}\|\Lambda\|_t - \beta \mathrm{d}t, \end{aligned}$$

where $\beta > 0$ and $\gamma \neq 0$ are constants, W is a standard Brownian motion and $\|\Lambda\|$ is the total variation of our control Λ that we require to be an adapted process of finite variation. The problem, motivated by a financial practice of hedging under transaction costs, is to minimize

$$\limsup_{T\to\infty} \frac{1}{T} E[Y_T^2] = \limsup_{T\to\infty} \frac{1}{T} E[\int_0^T (X_t^2 - 2\beta Y_t)\mathrm{d}t + 2\int_0^T Y_t \mathrm{d}\|\Lambda\|_t].$$

This is a 2 dimensional stochastic control problem which is degenerate (both the Brownian motion W and the control Λ are only one dimensional). The optimal control is, by a formal dynamic programming principle, expected to be a singular one which keeps (X, Y) inside a region; however we have not yet had satisfactory results both from theoretical and practical points of view for this original problem. In this talk, we focus on a restricted class of control

$$\mathrm{d}\Lambda = -\mathrm{sgn}(X_t)\gamma^2 c(X_t)\mathrm{d}t + \mathrm{d}L_t - \mathrm{d}R_t,$$

where c is a nonnegative continuous even function on an interval $[-b, b]$, $b > 0$ and L and R are nondecreasing processes with

$$\mathrm{d}L_t = 1_{\{X_t=-b\}}\mathrm{d}L_t, \quad \mathrm{d}R_t = 1_{\{X_t=b\}}\mathrm{d}R_t \tag{1}$$

which keep X stay in $[-b, b]$. Now our control is (b, c). The idea of the control is to push X towards 0. The absolutely continuous part of Λ determined by c pushes X regularly towards 0. The other parts, that turn out to be singularly continuous, are active only when X reaches the boundary of $[-b, b]$ and push X to prevent it from going out of the interval. Such a control exists; in fact, there exists a pathwise unique strong solution (X, L, R) of a Skorokhod SDE

$$\mathrm{d}X_t = \gamma \mathrm{d}W_t - \mathrm{sgn}(X_t)\gamma^2 c(X_t)\mathrm{dt} + \mathrm{d}L_t - \mathrm{d}R_t$$

on $[-b, b]$ when $x \mapsto -\mathrm{sgn}(x)c(x)$ is one-sided Lipschitz. The control Λ is then well-defined by (1). The optimal control in this restricted class is probably suboptimal for the original problem; however it has a certain advantage in its easy implementation. Also this type of control strategies has appeared in a related context of optimal hedging. Now, we are no more sure about the validity of the dynamic programming principle due to the constraint that the control Λ can only refer to the spot value of X. Although we can formally derive an HJB type equation, it is far from a standard form and difficult to solve. Here we present our results based on a probabilistic approach.

Theorem:

1. As $T \to \infty$,
$$\frac{1}{\sqrt{T}}(Y_T - \delta^{b,c}T) \to \mathcal{N}(0, Q^{b,c})$$
in law, where
$$\delta^{b,c} = \frac{\gamma^2}{a} - \beta, \;\; a = 2\int_0^b g(x)\mathrm{d}x, \;\; g(x) = \exp\left\{-2\int_0^x c(y)\mathrm{d}y\right\}$$
and
$$Q^{b,c} = \frac{2}{a}\int_0^b (x - \gamma h(x))^2 g(x)\mathrm{d}x, \;\; h(x) = \frac{2}{g(x)}\int_0^x \left(c(y) - \frac{1}{a}\right) g(y)\mathrm{d}y.$$

2. $$\lim_{T\to\infty} \frac{1}{T} E[(Y_T - \delta^{b,c}T)^2] = Q^{b,c}.$$

3. $$\inf_{\delta^{b,c}=0} Q^{b,c} = \gamma^2 \eta\left(\frac{\gamma}{\beta}\right),$$
where
$$\eta(x) = \begin{cases} 0 & \text{if } -2 < x \le 1, \\ \frac{4}{3}\frac{(x+2)^2(x-1)}{x^3(4-x)} & \text{if } 1 < x < 2, \\ \frac{1}{12}(x+2)^2 & \text{if } |x| \ge 2. \end{cases}$$

The proof of the convergences is not difficult. The mathematically challenging part is the minimization of $Q^{b,c}$. We can give an explicit sequence of controls (b_n, c_n) with $\delta^{b_n,c_n} = 0$ such that Q^{b_n,c_n} converges to the infimum. In fact, $b_n = \gamma^2/2\beta$ and $c_n = 0$ when $|\gamma| \ge 2\beta$. When $|\gamma| < 2\beta$ on the other hand, $b_n \to \infty$ as $n \to \infty$ and the pointwise limit of $c_n(x)$ is given by
$$c_\infty(x) = \frac{\gamma + 2\beta}{2(\gamma - \beta l)} \frac{1}{\gamma + |x|} 1_{\{|x| \ge l\}}, \;\; l = \frac{2(\gamma - \beta)_+}{4\beta - \gamma}.$$
The key for the minimization is to show
$$\inf_{y \in \mathcal{Y}_a} \int_0^1 \left(y(u) + \gamma + \frac{2\gamma}{a}(u-1)y'(u)\right)^2 \mathrm{d}u = \gamma^2 \eta(a/\gamma),$$
where $\mathcal{Y}_a$ is the set of the convex functions on $[0, 1]$ with $y(0) = 0$ and $y'(0) = a/2$.