Symposium on Stochastic Analysis and its Applications

Osaka University, October 20–22, 2015

On Convergence of Environment-Dependent Models

Isamu DÔKU

Department of Mathematics, Saitama University

1. Summary

We consider an environment-dependent spatial model.

This random model is related to the stochastic interacting system.

We shall show that rescaled processes converge to a Dawson-Watanabe superprocess.

Formulation is due to setup of measure-valued branching Markov processes.

The first step toward a transformation of model into a superprocess is based upon construction of empirical measures.

2. Introduction

We shall introduce an environment-dependent random model.

Let $\mathbb{Z}^d$ be a d-dimensional integer lattice.

Suppose that each site on $\mathbb{Z}^d$ is occupied by all means by either one of the two species.

At each random time, a particle dies and is replaced by a new one, but the random time and the type chosen of the species are assumed to be determined by the environment conditions around the particle.

3. Random Function

The random function $\eta_t : \mathbb{Z}^d \to \{0, 1\}$ denotes the state at time t, and each number of $\{0, 1\}$ denotes the label of the type chosen of the two species. When we set $\|y\|_\infty := \max_i y_i$ for $y = (y_1, \dots, y_d)$, then we define

$$\mathcal{N}_x := x + \{y : \; 0 < \|y\|_\infty \leqslant r\}, \tag{1}$$

where r is a positive constant. For $i = 0, 1$, let $f_i(x, \eta)$ be a frequency of appearance of type i in the neighborhood $\mathcal{N}_x$ of x for η. In other words,

$$f_i(x) \equiv f_i(x, \eta) := \frac{\#\{y : \; \eta_t(y) = i \; ; \; y \in \mathcal{N}_x\}}{\#\mathcal{N}_x}. \tag{2}$$

4. Dynamics of States

For non-negative parameters $\alpha_{ij} \geq 0$, the dynamics of η_t is defined as follows. The state η makes transition $0 \to 1$ at rate

$$\frac{\lambda f_1(f_0 + \alpha_{01} f_1)}{\lambda f_1 + f_0}, \tag{3}$$

and it makes transition $1 \to 0$ at rate

$$\frac{f_0(f_1 + \alpha_{10} f_0)}{\lambda f_1 + f_0}. \tag{4}$$

5. Interpretation of the Rate

The particle of type i dies at rate $f_i + \alpha_{ij} f_j$, and is replaced instantaneously by either one of the two species chosen at random, according to the proliferation rate of type 0 and the interaction (= the competitive result) with the particle of type 1. The density-dependent death rate $f_i + \alpha_{ij} f_j$ consists of the intraspecific and interspecific competitive effects. We assume that competitive two species possess the same intensity of intraspecific interaction. The exchange of particles after death is described in the form being proportional to the weighted density between the two species, expressed by a parameter λ. Assume that $\lambda \geq 1$. The case of $\lambda = 1$ means that the contribution to a local appearance rate between the two competitive species is equivalent. When $\lambda \geq 1$, then it means that the type 1 has a higher proliferation rate than the type 0. In this talk we shall discuss some convergence result of the environment-dependent spatial models.

6. Scaling Rule

For brevity's sake we shall treat a simple case $\lambda = 1$ only below.

For $N = 1, 2, \ldots$,

let $m_N \in \mathbb{N}$, and we put $\ell_N := m_N\sqrt{N}$, and $\mathbb{S}_N := \mathbb{Z}^d/\ell_N$,

and $W_N = (W_N^1, \ldots, W_N^d) \in (\mathbb{Z}^d/M_N) \setminus \{0\}$ is defined as a random vector satisfying

(i) $\mathcal{L}(W_N) = \mathcal{L}(-W_N)$;

(ii) $E(W_N^i W_N^j) \to \delta_{ij}\sigma^2 (\geq 0)$ (as $N \to \infty$);

(iii) $\{|W_N|^2\}$ ($N \in \mathbb{N}$) is uniformly integrable.

Here $\mathcal{L}(Y)$ indicates the law of a random variable Y.

7. Rescaled Process

For the kernel $p_N(x) := P(W_N/\sqrt{N} = x)$, $x \in \mathbb{S}_N$ and $\eta \in \{0,1\}^{\mathbb{S}_N}$, we define the scaled frequency f_i^N as

$$f_i^N(x,\eta) = \sum_{y \in \mathbb{S}_N} p_N(y-x) 1_{\{\eta(y)=i\}}, \qquad (i = 0, 1). \tag{5}$$

We denote by η_t^N the state determined by the scaled frequency depending on α_i^N and p_N. As a matter of fact, the rescaled process $\eta_t^N : \mathbb{S}_N \ni x \mapsto \eta_t^N(x) \in \{0,1\}$ is determined by the following state transition law, nemaly, it makes transition $0 \to 1$ at rate $N f_1^N(f_0^N + \alpha_0^N f_1^N)$, or else it makes transition $1 \to 0$ at rate $N f_0^N(f_1^N + \alpha_1^N f_0^N)$. We denote the rescaled process η_t^N by the symbol $Res(p_N, \alpha_i^N)$.

8. Empirical Measure

On this account, we may define the associated measure-valued process (or its corresponding empirical measure) as

$$X_t^N := \frac{1}{N} \sum_{x \in \mathbb{S}_N} \eta_t^N(x) \delta_x. \tag{6}$$

For the initial value X_0^N, we assume that

$$\sup_N \langle X_0^N, 1 \rangle < \infty, \qquad X_0^N \to X_0 \quad \text{in} \quad M_F(\mathbb{R}^d) \quad (N \to \infty), \tag{7}$$

where $M_F(\mathbb{R}^d)$ is the totality of all the finite measures on $\mathbb{R}^d$, equipped with the topology of weak convergence. For a finite measure $\mu \in M_F(E)$ with a topological space E, we use the notation $\langle \mu, \varphi \rangle = \int_E \varphi(x) \mu(dx)$ for integral of a measurable function φ over E with respect to a measure μ on E. Note that the convergence in (7) is that in the sense of weak convergence for measures.

9. Variational Derivative

Let $\Omega_D := D([0,\infty), M_F(\mathbb{R}^d))$ be the Skorokhod space of all the $M_F(\mathbb{R}^d)$-valued cadlag paths, and $\Omega_C := C([0,\infty), M_F(\mathbb{R}^d))$ be the space of all the $M_F(\mathbb{R}^d)$-valued continuous paths, equipped with uniform convergence topology on compacts. $C_b^\infty(\mathbb{R}^d)$ consists of the infinitely differentiable functions on $\mathbb{R}^d$ whose derivatives of any order k are bounded and continuous. On the other hand, the first order variational derivative of a function F on $M_F(E)$ relative to $\mu \in M_F(E)$ is defined as

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{r \to 0+} \frac{F(\mu + r \cdot \delta_x) - F(\mu)}{r}, \qquad (x \in E) \tag{8}$$

if the limit in the right-hand side of (8) exists. In addition, the second order variational derivative $\delta^2 F(\mu)/\delta\mu(x)^2$ is defined as the first order variational derivative of $G(\mu) = \delta F(\mu)/\delta\mu(x)$ if its limit exists.

10. Generator of Superprocess

We define the generator $\mathcal{L}_0$ as

$$\mathcal{L}_0 F(\mu) := \int_E A \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) + \int_E \gamma \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx), \tag{9}$$

where $A[\cdot] = \frac{\sigma^2}{2}\Delta[\cdot] + \theta[\cdot]$ and $\gamma > 0$. If $M_F(E)$-valued continuous stochastic process $X = \{X_t, P_\eta\}$ is a solution to the $(\mathcal{L}_0, \mathrm{Dom}(\mathcal{L}_0))$-martingale problem, then $X = \{X_t, P_\eta\}$ is called a Dawson-Watanabe superprocess, or DW superprocess in short, where $2\gamma \geq 0$ is a branching rate, $\theta \in \mathbb{R}$ is a drift term and $\sigma^2 > 0$ is a diffusion coefficient. More precisely, $X_0 = \eta \in M_F(E)$ holds P_η-a.s., and for any function $F = F(\mu) \in \mathrm{Dom}(\mathcal{L}_0)$ defined on $M_F(E)$,

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}_0 F(X_s) ds \tag{10}$$

is a P_η-martingale.

11. Assumptions (1)

Let $\{\xi_t^x\}$ be a continuous time random walk with rate N and step distribution p_N starting at a point $x \in \mathbb{S}_N$, and $\{\hat{\xi}_t^x\}$ be a continuous time coalescing random walk with rate N and step distribution p_N starting at a point x. For a finite set $A \subset \mathbb{S}_N$, we denote by $\tau(A)$ the time when all the particles starting from A finally coalesce into a single particle, that is to say, we define

$$\tau(A) := \inf\left\{ t > 0 : \ \#\{\hat{\xi}_t^x;\ x \in A\} = 1 \right\}. \tag{11}$$

Take a sequence $\{\varepsilon_N\}$ of positive numbers such that $\varepsilon_N \to 0$ and $N\varepsilon_N \to \infty$ as $N \to \infty$.

12. Assumption (2)

Moreover, we suppose that when $N \to \infty$,

$$N \cdot P(\xi^0_{\varepsilon_N} = 0) \to 0 \qquad \text{and} \tag{12}$$

$$\sum_{e \in \mathbb{S}_N} p_N(e) \cdot P(\tau(\{0, e\}) \in (\varepsilon_N, t]) \to 0 \quad (\forall t > 0). \tag{13}$$

We also assume now that the following limits exist :

$$\lim_{N \to \infty} \sum_{e \in \mathbb{S}_N} p_N(e) \cdot P(\tau(\{0, e\}) > \varepsilon_N) = \exists \gamma (> 0) \tag{14}$$

$$\text{and} \qquad \lim_{N \to \infty} P\left(\tau(A/\ell_N) \leqslant \varepsilon_N\right) = \exists \zeta(A) \tag{15}$$

holds for any finite subset $A \subset \mathbb{Z}^d$. And also we denote by S_F the totality of all the finite subsets in $\mathbb{Z}^d$.

13. Perturbation (1)

According to Liggett (1985), we consider decomposing proper components of our model $Res(p_N, \alpha_i^N)$ into two parts; a part of the principal interacting particle system and the other part. Based upon the notation in Liggett (1999), we consider decomposing the rate function $c_N(x, \eta)$. In fact, we shall rewrite first a rate $Nf_i^N(f_j^N + \alpha_j^N f_i^N)$ into a new rate $Nf_i^N + \theta_j^N (f_i^N)^2$ by using a relation $\theta_i^N = N(\alpha_i^N - 1)$, and next decompose the rate function $c_N(x, \eta)$ (which changes the coordinate $\eta(x)$ into $1 - \eta(x)$) as

$$c_N(x, \eta) = N \cdot c_0(x, \eta) + c_p(x, \eta) \geq 0, \tag{16}$$

14. Perturbation (2)

where

$$c_0(x,\eta) := \sum_{e\in\mathbb{S}_N} p_N(e)1_{\{\eta(x+e)\neq\eta(x)\}}, \qquad \text{and} \tag{17}$$

$$c_p(x,\eta) := \theta_0^N(f_1^N(x,\eta))^2 1_{\{\eta(x)=0\}} + \theta_1^N(f_0^N(x,\eta))^2 1_{\{\eta(x)=1\}} \tag{18}$$

$$= \sum_{A\in S_F}\left(\prod_{e\in A/\ell_N}\eta(x+e)\right)(\beta_N(A)1_{\{\eta(x)=0\}} + \delta_N(A)1_{\{\eta(x)=1\}}).$$

On the assumption that for real-valued functions β_N and δ_N defined on S_F, there exist proper real-valued functions β and δ defined on S_F such that $\beta_N \to \beta$ and $\delta_N \to \delta$ are valid for each point of S_F as $N \to \infty$, we consider the convergence of the law of the empirical measure $X_\cdot^N$.

15. An Estimate (1)

For simplicity, when we set

$$F_1(S_F) := \{f : S_F \to \mathbb{R};\ \|f\|_1 := \sum_{A \in S_F} |f(A)| < \infty\}, \tag{19}$$

then it follows that $\beta_N(\cdot)\zeta_N(\cdot) \to \beta(\cdot)\zeta(\cdot)$ in $F_1(S_F)$ as $N \to \infty$. Under these circumstances, we have

$$\sup_N \sum_{A \in S_F} \max(\#\{A\}, 1)(|\beta_N(A)| + |\delta_N(A)|) < \infty \tag{20}$$

16. An Estimate (2)

and the following estimate holds: i.e., for a certain positive constant $C(\delta) > 0$,

$$\sum_{y \in \mathbb{Z}^d} p_N(y/\ell_N)(\eta(y) - 1) \leqslant C(\delta) \sum_{A \in S_F} \delta_N(A) \prod_{a \in A} \eta(a) \tag{21}$$

holds. While, when we define

$$\theta^1(\beta, \zeta(\cdot)) := \sum_{A \in S_F} \beta(A)\zeta(A) \qquad \text{and} \tag{22}$$

$$\theta^2(\beta, \delta, \zeta(\cdot)) := \sum_{A \in S_F} (\beta(A) + \delta(A))\zeta(A \cup \{0\}), \tag{23}$$

then we put $\theta = \theta^1(\beta, \zeta(\cdot)) - \theta^2(\beta, \delta, \zeta(\cdot))$.

17. Convergence Result

THEOREM 1. (Convergence) When we denote the law of a measure-valued stochastic process $X_\cdot^N$ on the path space Ω_D by P_N, then there exists a probability measure $P^* \in \mathcal{P}(\Omega_C)$ such that

$$P_N \implies P^*_{X_0} \quad (\text{as} \quad N \to \infty). \tag{24}$$

Then there exists a $M_F(\mathbb{R}^d)$-valued stochastic process $X_t = X_t^{2\gamma,\theta,\sigma^2}$ named a DW superprocess with parameters $2\gamma > 0$, $\theta \in \mathbb{R}$ and $\sigma^2 > 0$, satisfying that X_t^N converges to $X_t^{2\gamma,\theta,\sigma^2}$ as $N \to \infty$ in the sense of weak convergence for measures, and $P^*_{X_0}$ is the law of $X_t^{2\gamma,\theta,\sigma^2}$.

18. Martingale Problem

It is interesting to note that the DW superprocess $(X_t, P^*_{X_0})$ that appears in the limit gives a solution to the following martingale problem. Namely, $X_0 = \mu$ holds $P^*_{X_0}$-a.s., and for $f \in C^2(\mathbb{R})$ and $\varphi \in \mathrm{Dom}(A)$

$$f(\langle X_t, \varphi\rangle) - f(\langle \mu, \varphi\rangle) - \int_0^t \frac{\sigma^2}{2} f'(\langle X_s, \varphi\rangle) \cdot \langle X_s, \Delta\varphi\rangle ds$$
$$- \theta \int_0^t f'(\langle X_s, \varphi\rangle) \cdot \langle X_s, \varphi\rangle ds - \gamma \int_0^t f''(\langle X_s, \varphi\rangle) \cdot \langle X_s, \varphi^2\rangle ds \tag{25}$$

is a $P^*_{X_0}$-martingale.

19. Sketch of Proof: Step 1 (1)

Step 1. In this section we shall introduce a sketch of proof of our main result Theorem 1, which asserts that rescaled empirical measures related to our environment-dependent spatial models converge to a Dawson-Watanabe superprocess in the sense of weak topology under suitable conditions. First of all, note that our basic setup yields to the finiteness of

$$E[\sup_{0\leqslant t\leqslant T} |\eta_t^N|^2] < \infty, \qquad \forall T > 0. \tag{26}$$

Based upon the above-mentioned estimation, combining the discussion on death and birth processes to a series of results for voter models together, the following first decomposition for rescaled process models $Res(p_N, \alpha_i^N)$:

$$\eta_t^N(x) = \eta_0^N(x) + M_t^{N,x} + D_t^{N,x}, \qquad \forall x \in \mathbb{S}_N;\, t \geq 0, \tag{27}$$

20. Sketch of Proof: Step 1 (2)

where $M_t^{N,x}$ is a square integrable orthogonal martingale, and its predictable quadratic variation process is given by

$$\langle M^{N,x}\rangle_t = \int_0^t \left\{ \sum_{y\in\mathbb{S}_N} N\cdot p_N(y-x)(\xi_s^N(y)-\xi_s^N(x))^2 + \sum_{A\in S_F}\left(\prod_{e\in A/\ell_N}\xi_s^N(x+e)\right)(\beta_N(A)1_{\{\xi_s^N(x)=0\}}+\delta_N(A)1_{\{\xi_s^N(x)=1\}})\right\} ds. \tag{28}$$

Moreover, the term $D_t^{N,x}$ is given by

$$D_t^{N,x} = \int_0^t \left\{ \sum_{y\in\mathbb{S}_N} N\cdot p_N(y-x)(\xi_s^N(y)-\xi_s^N(x)) + \sum_{A\in S_F}\left(\prod_{e\in A/\ell_N}\xi_s^N(x+e)\right)(\beta_N(A)1_{\{\xi_s^N(x)=0\}}-\delta_N(A)1_{\{\xi_s^N(x)=1\}})\right\} ds. \tag{29}$$

21. Sketch of Proof: Step 1 (3)

Next, by employing Itô's formula to $f(\eta; x, y) := \eta(x)\eta(y)$, we may apply the decomposition theorem for semimartingales to η_t^N to obtain

$$\eta_t^N(x) = \eta_0^N(x) + 2\int_0^t \eta_{s-}^N(x) dD_s^{N,x} + 2\int_0^x \eta_{s-}^N(x) dM_s^{N,x} + [M^{N,x}]_t, \qquad (30)$$

where $[M^{N,x}]_t$ is the quadratic variation function for martingale $M_t^{N,x}$, and the term $[M^{N,x}]_t - \langle M^{N,x}\rangle_t$ becomes a martingale. And also the integral term $\int_0^t \eta_{s-}^N(x) dM_s^{N,x}$ is a stochastic integral of Itô type with respect to a square integrable martingale, which itself turns out to be a martingale again. Once this form (30) can be derived, stochastic analysis is easily applicable to the object, with the result that we can derive with ease the decomposition of measure-valued process X_t^N which just corresponds to our original spatial model $Res(p_N, \alpha_i^N)$.

22. Sketch of Proof: Step 1 (4)

As a matter of fact, for any $\varphi \in C_b([0,T] \times \mathbb{S}_N)$ and $0 \leqslant t \leqslant T$, X_t^N permits the following second decomposition

$$\langle X_t^N, \varphi_t \rangle = \langle X_0^N, \varphi_0 \rangle + D_t^N(\varphi) + M_t^N(\varphi), \tag{31}$$

where $M_t^N(\varphi)$ is a square integrable martingale, and its predictable quadratic variation process $\langle M^N(\varphi) \rangle_t$ is also concretely expressed by the principal components of the model $Res(p_N, \alpha_i^N)$, and moreover, it is uniquely determined as well. More precisely, for $\psi \in C_b(\mathbb{S}_N)$, we put

$$F_1(\psi) := \sum_{y \in \mathbb{S}_N} N \cdot p_N(y-x)(\psi(y) - \psi(x)), \qquad F_2(\psi, A) := \prod_{e \in A/\ell_N} \psi(x+e),$$

$$F_3(\psi) := (\psi(y) - \psi(x))^2 \quad \text{and} \quad F_4(\psi, A) := \beta_N(A) 1_{\{\psi(x)=0\}} + \delta_N(A) 1_{\{\psi(x)=1\}}.$$

23. Sketch of Proof: Step 1 (5)

Here for $\varphi_s(x) \equiv \varphi(s,x) \in C_b([0,T] \times \mathbb{S}_N)$ and $\dot{\varphi}_s(x) \equiv \frac{\partial}{\partial s}\varphi(s,x) \in C_b([0,T] \times \mathbb{S}_N)$, $D_t^N(\varphi)$ can also be decomposed as follows:

$$D_t^N(\varphi) = D_t^{N,1}(\varphi) + D_t^{N,2}(\varphi) + D_t^{N,3}(\varphi). \tag{32}$$

We have respectively

$$D_t^{N,1}(\varphi) := \int_0^t X_s^N(F_1(\varphi_s) + \dot{\varphi}_s)ds, \tag{33}$$

$$D_t^{N,2}(\varphi) := \frac{1}{N}\int_0^t \sum_{x\in\mathbb{S}_N} \varphi_s(x) \sum_{A\in S_F} \beta_N(A)F_2(\eta_s^N, A)ds, \tag{34}$$

$$D_t^{N,3}(\varphi) := \frac{(-1)}{N}\int_0^t \sum_{x\in\mathbb{S}_N} \varphi_s(x) \sum_{A\in S_F} (\beta_N(A) + \delta_N(A)) \cdot \eta_s^N(x)F_2(\eta_s^N, A)ds. \tag{35}$$

24. Sketch of Proof: Step 1 (6)

While, $M_t^N(\varphi)$ is a martingale which is given as a stochastic integral

$$M_t^N(\varphi) = \frac{1}{N} \sum_{x \in \mathbb{S}_N} \int_0^t \varphi_s(x) dM_s^{N,x} \tag{36}$$

by making use of the martingale term $M_t^{N,x}$ in the first decomposition (27). By virtue of the finiteness of $E[\sup_{t \leqslant T} |\eta_t^N|^2] < \infty$, it follows that

$$E\left(\sum_{x \in \mathbb{S}_N} \left\langle \int_0^{\cdot} \varphi_s(x) dM_s^{N,x} \right\rangle_T \right) < \infty \tag{37}$$

holds, hence it is clear that the series (36) should be convergent in L^2 uniformly relative to $t (\leqslant T)$. Thus we can deduce that the martungale $M_t^N(\varphi)$ is L^2-integrable.

25. Sketch of Proof: Step 1 (7)

Furthermore, we observe that the predictable quadratic variation process $\langle M^N(\varphi)\rangle_t$ of $M_t^N(\varphi)$ consists of the following two terms:

$$\langle M^N(\varphi)\rangle_t = \langle M^N(\varphi)\rangle_t^1 + \langle M^N(\varphi)\rangle_t^2, \tag{38}$$

and each term is given by

$$\langle M^N(\varphi)\rangle_t^1 := \frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{S}_N} \varphi_s^2(x) \sum_{y\in\mathbb{S}_N} N\cdot p_N(y-x)F_3(\xi_s^N)ds, \tag{39}$$

$$\langle M^N(\varphi)\rangle_t^2 := \frac{1}{N^2}\int_0^t \sum_{x\in\mathbb{S}_N} \varphi_s^2(x) \sum_{\Lambda\in S_F} F_2(\xi_s^N, A)F_4(\xi_s^N, A))ds \tag{40}$$

respectively.

26. Second Step

Step 2. Since we are going to discuss the convergence problem for the rescaled process constructed in the previous step, when we denote the law of measure-valued process X^N on the path space Ω_D by the symbol $P_N \in \mathcal{P}(\Omega_D)$, then we consider next the tightness of a family of probability measures $\{P_N;\ N \geq 1\}$ on the path space Ω_C. Recall that when E is a Polish space, the necessary and sufficient condition for a sequence of probability measures $\{P_n\}$ on the Skorokhod space $D([0,\infty), E)$ to be C-tight is that $\{P_n\}$ is itself tight, and also that the measure support of all the limit points (= the limit probability measures) lies on the space of continuous paths $C([0,\infty), E)$. On the other hand, thanks to the Prokhorov theorem, we know that a sequence of laws on the path space $\{P_n\}$ ($P_n \in \mathcal{P}(\Omega_D)$) is tight if and only if $\{P_n\}$ is relatively compact. Therefore, by resorting to the Jakubowski theorem for weak convergence in Ω_D, we can easily derive the C-tightness of the family $\{P_N, N \in \mathbb{N}\}$. Hence, we finally prove that there exists a proper subsequence $\{P_{N(k)}\}$ such that $P_{N(k)}$ converges weakly to a probability measure $P_0 \in \mathcal{P}(\Omega_C)$.

27. Third Step (1)

The second decomposition has also an integral form

$$\langle X_t^N, \varphi_t \rangle = \langle X_0^N, \varphi_0 \rangle + \int_0^t \tilde{D}_s^N(\varphi)ds + \int_0^t dM_s^N(\varphi) \qquad (41)$$

where

$$D_t^N(\varphi) = \int_0^t \tilde{D}_s^N(\varphi)ds = \int_0^t \tilde{D}_s^{N,1}(\varphi)ds + \int_0^t \tilde{D}_s^{N,2}(\varphi)ds + \int_0^t \tilde{D}_s^{N,3}(\varphi)ds$$

For $f \in C^2(\mathbb{R})$, we may apply Itô's formula to (41) to obtain

$$f(\langle X_t^N, \varphi_t \rangle) = f(\langle X_0^N, \varphi_0 \rangle) + \int_0^t f'(\langle X_s^N, \varphi_s \rangle)\tilde{D}_s^N(\varphi)ds$$

$$+ \int_0^t f'(\langle X_s^N, \varphi \rangle)dM_s^N(\varphi) + \frac{1}{2}\int_0^t f''(\langle X_s^N, \varphi \rangle)d\langle M^N(\varphi)\rangle_s \quad (42)$$

28. Third Step (2)

For $k = 1, 2, 3$, we put $D_t^{N,k}(\varphi) := \int_0^t \tilde{D}_s^{N,k}(\varphi) ds$, and we also use a new symbol $\tilde{A} := A - \theta$ for simplicity. Moreover, we define

$$\begin{aligned} \hat{D}_t^{N,1}(\varphi) &:= \int_0^t f'(X_s^N(\varphi))\tilde{D}_s^{N,1}(\varphi)ds \\ \hat{D}_t^{N,2}(\varphi) &:= \int_0^t f'(X_s^N(\varphi))\tilde{D}_s^{N,2}(\varphi)ds \\ \hat{D}_t^{N,3}(\varphi) &:= \int_0^t f'(X_s^N(\varphi))\tilde{D}_s^{N,3}(\varphi)ds \end{aligned} \tag{43}$$

Similarly, we put $\langle \hat{M}^{N,k}(\varphi)\rangle_t := \int_0^t f''(X_s^N(\varphi)) d\langle M^N(\varphi)\rangle_s^k$ for $k = 1, 2$.

29. Third Step (3)

Then it is easy to see that, as $N \to \infty$,

$$E\left|\hat{D}_t^{N,1}(\varphi) - \int_0^t f'(X_s^N(\varphi))\langle X_s^N, (\tilde{A} + \frac{\partial}{\partial s})\varphi\rangle ds\right|^2 \to 0 \tag{44}$$

$$E\left|\hat{D}_t^{N,2}(\varphi) + \hat{D}_t^{N,3}(\varphi) - \int_0^t f'(X_s^N(\varphi))\langle X_s^N, \theta\varphi_s\rangle ds\right|^2 \to 0 \tag{45}$$

$$E\left|\langle \hat{M}^{N,1}(\varphi)\rangle_t + \langle \hat{M}^{N,2}(\varphi)\rangle_t - \int_0^t f''(X_s^N(\varphi))\langle X_s^N, 2\gamma\varphi_s^2\rangle ds\right|^2 \to 0 \tag{46}$$

When we set $\hat{D}_t^N(\varphi) = \hat{D}_t^{N,1}(\varphi) + \hat{D}_t^{N,2}(\varphi) + \hat{D}_t^{N,3}(\varphi)$, then an application of (44) and (45) yields to

$$P\left(\left|\hat{D}_t^N(\varphi) - \int_0^t f'(X_s^N(\varphi))\langle X_s^N, (\tilde{A} + \partial_s)\varphi\rangle ds - \int_0^t f'(X_s^N(\varphi))\langle X_s^N, \theta\varphi_s\rangle ds\right| > \varepsilon\right) \to 0$$

30. Third Step (4)

Then (46) yields to

$$P\left(\left|\langle \hat{M}^{N,1}(\varphi)\rangle_t + \langle \hat{M}^{N,2}(\varphi)\rangle_t - \int_0^t f''(X_s^N(\varphi))\langle X_s^N, 2\gamma\varphi_s^2\rangle ds\right| > \varepsilon\right) \to 0$$

Consequently, when the subsequence convergence

$$P(X_t^{N(k)} \in (\cdot)) \quad \Longrightarrow \quad P(X_t \in (\cdot)) \qquad \text{in} \quad D([0,\infty), M_F(\mathbb{R}^d)) \tag{47}$$

holds for a measure-valued process $X_t(\omega)$, then we observe that a triplet $(X_t^{N(k)}, \hat{D}_t^{N(k)}(\varphi), \langle \hat{M}_{(\varphi)}^{N(k),1}\rangle_t + \langle \hat{M}_{(\varphi)}^{N(k),2}\rangle_t)$ is C-tight in the Skorokhod space

$$D([0,\infty), M_F(\mathbb{R}^d) \times C(\mathbb{R}) \times C(\mathbb{R}^+))$$

Hence, by virtue of application of Skorokhod theorem for sub-subsequence, it follows finally that

$$(X_t^{N(k)'}, \hat{D}_t^{N(k)'}(\varphi), \langle \hat{M}_{(\varphi)}^{N(k)',1}\rangle_t + \langle \hat{M}_{(\varphi)}^{N(k)',2}\rangle_t) \quad \longrightarrow \quad (X_t, \hat{D}_t(\varphi), Q_t(\varphi)) \quad \text{a.s.}$$

31. Third Step (5)

Thus we attain that

$$\int_0^t f'(X_s(\varphi))dM_s(\varphi) = f(\langle X_t, \varphi\rangle) - f(\langle X_0, \varphi\rangle)$$
$$- \int_0^t f'(X_s(\varphi))\langle X_s, A\varphi\rangle ds - \int_0^t f''(X_s(\varphi))\langle X_s, \gamma\varphi^2\rangle ds \qquad (48)$$

is a continuous, $\mathcal{F}_t^X$-measurable, L^2-martingale.

Equivalently, for $F(\mu) = f(\langle \mu, \varphi\rangle)$ with $F \in \mathrm{Dom}(\mathcal{L}_0)$,

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}_0 F(X_s)ds \quad \text{is a} \quad P^*_{X_0} - \text{martingale}$$

As a consequence, it is proven that the law $P(X_\cdot \in (\cdot))$ of the limit process $X = \{X_t\}$ satisfies the martingale problem characterizing $P^*_{X_0} \in \mathcal{P}(\Omega_C)$.

32. Another Situation (1)

Assume that $\exists\{\varepsilon_N^*\}$, $\varepsilon_N^* > 0$ such that $\varepsilon_N^* \to 0$, $N \cdot \varepsilon_N^* \to \infty$ (as $N \to \infty$), and

$$\gamma_N(x) = \sum_{e \in \mathbb{S}_N} p_N(e,x) \cdot P(\hat{\tau}^N(\{0,e\}) > \varepsilon_N^*) \tag{49}$$

and $\gamma_N(x) \to \gamma(x)$ as $N \to \infty$. Here $\hat{\tau}^N(A)$ denotes the time at which all particles starting from a set $A \subset \mathbb{S}_N$ have coalesced into a single particle.

For a bounded and measurable function $\phi : [0,T] \times \mathbb{S}_N \to \mathbb{R}$, there is a constant $C > 0$ such that

$$\langle M^N(\phi)\rangle_t^1 = 2\int_0^t X_s^N(\phi_s^2 f_0^N(\eta_s^N))ds + \int_0^t \Xi_s^N(\phi_s)ds \tag{50}$$

$$\text{(Estimate (1))} \qquad |\Xi_s^N(\phi_s)| \leqslant \frac{C\|\phi\|^2}{\sqrt{N}}\langle X_s^N, 1\rangle \tag{51}$$

$$\text{(Estimate (2))} \qquad |\langle M^N(\varphi)\rangle_t^2| \leqslant \frac{C\|\phi\|^2}{N}\int_0^t \langle X_s^N, 1\rangle ds \tag{52}$$

33. Another Situation (2)

As a result that

$$E\left|\int\{X_s^N(\phi_s^2 f_1^N(\eta_s^N)) - X_s^N((1-\gamma_N(x))\phi_s^2)\}ds\right|^2 \to 0 \qquad (N\to\infty) \tag{53}$$

Therefore it follows immediately that

$$E\left|\langle M^N(\varphi)\rangle_t^1 + \langle M^N(\varphi)\rangle_t^2 - 2\int_0^t \langle X_s^N, \gamma(x)\phi_s^2\rangle ds\right|^2 \to 0 \qquad (N\to\infty) \tag{54}$$

We define

$$\mathcal{L}_1 F(\mu) := \int_E A\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) + \int_E \gamma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx) \tag{55}$$

where $F \in \mathrm{Dom}(\mathcal{L}_1)$ and $A = \frac{\sigma^2}{2}\Delta + \theta$.

34. Another Situation (3)

Theorem 2. (Convergence) When we denote the law of a measure-valued stochastic process X^N on the path space Ω_D by P_N, then there exists a probability measure $\hat{P} \in \mathcal{P}(\Omega_C)$ such that

$$P_N \quad \Longrightarrow \quad \hat{P}_{X_0} \qquad (\text{as} \quad N \to \infty). \tag{1}$$

Then there exists a $M_F(\mathbb{R}^d)$-valued stochastic process $X_t = X_t^{\gamma(x)}$ named a superprocess with spatially dependent parameter $\gamma(x) > 0$, satisfying that X_t^N converges to $X_t^{\gamma(x)}$ as $N \to \infty$ in the sense of weak convergence for measures, and $\hat{P}_{X_0}$ is the law of $X_t^{\gamma(x)}$ with the initial value X_0.