On Convergence of Environment-Dependent Models

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Let $\mathbb{Z}^d$ be a d-dimensional lattice, and each site on $\mathbb{Z}^d$ is occupied by either one of the two species. At each random time, a particle dies and is replaced by a new one, but the random time and the type chosen of the species are assumed to be determined by the environment conditions around the particle. The random function $\eta_t : \mathbb{Z}^d \to \{0,1\}$ denotes the state at time t, and each number of $\{0,1\}$ denotes the label of the type chosen of the two species. When we set $\|y\|_\infty := \max_i y_i$, we define $\mathcal{N}_x := x + \{y : \ 0 < \|y\|_\infty \leqslant r\}$. For $i = 0, 1$, let $f_i(x, \eta)$ be a frequency of type i in the neighborhood $\mathcal{N}_x$ of x for η. For non-negative parameters $\alpha_{ij} \geq 0$, the dynamics of η_t is defined as follows. The state η makes transition $0 \to 1$ at rate $\lambda f_1(f_0 + \alpha_{01} f_1)/(\lambda f_1 + f_0)$, and it makes transition $1 \to 0$ at rate $f_0(f_1 + \alpha_{10} f_0)/(\lambda f_1 + f_0)$. The exchange of particles after death is described in the form being proportional to the weighted density between the two species, expressed by a parameter λ. For brevity's sake we shall treat a simple case $\lambda = 1$ only in what follows. For $N = 1, 2, \ldots$, let $m_N \in \mathbb{N}$, and we put $\ell_N := m_N \sqrt{N}$, and $\mathbb{S}_N := \mathbb{Z}^d/\ell_N$. While, $W_N = (W_N^1, \ldots, W_N^d) \in (\mathbb{Z}^d/m_N) \setminus \{0\}$ is defined as a random vector satisfying (i) $\mathcal{L}(W_N) = \mathcal{L}(-W_N)$; (ii) $E(W_N^i W_N^j) \to \delta_{ij}\sigma^2 (\geq 0)$ (as $N \to \infty$); (iii) $\{|W_N|^2\}$ $(N \in \mathbb{N})$ is uniformly integrable. Here $\mathcal{L}(Y)$ indicates the law of a random variable Y. For the kernel $p_N(x) := P(W_N/\sqrt{N} = x)$, $x \in \mathbb{S}_N$ and $\eta \in \{0,1\}^{\mathbb{S}_N}$, we define the scaled frequency f_i^N as

$$f_i^N(x, \eta) = \sum_{y \in \mathbb{S}_N} p_N(y - x) 1_{\{\eta(y) = i\}}, \qquad (i = 0, 1). \tag{1}$$

We denote by η_t^N the state determined by the scaled frequency depending on α_i^N and p_N. On this account, we may define the associated measure-valued process as

$$X_t^N := \frac{1}{N} \sum_{x \in \mathbb{S}_N} \eta_t^N(x) \delta_x. \tag{2}$$

For the initial value X_0^N, we assume that $\sup_N \langle X_0^N, 1 \rangle < \infty$ and $X_0^N \to X_0$ in $M_F(\mathbb{R}^d)$ as $N \to \infty$, where $M_F(\mathbb{R}^d)$ is the totality of all the finite measures on $\mathbb{R}^d$, equipped with the topology of weak convergence. Let $\{\xi_t^x\}$ be a continuous time random walk with rate N and step distribution p_N starting at a point $x \in \mathbb{S}_N$, and $\{\hat{\xi}_t^x\}$ be a continuous time coalescing random walk with rate N and step distribution p_N starting at a point x. For a finite set $A \subset \mathbb{S}_N$, we denote by $\tau(A)$ the time when all the particles starting from A finally coalesce into a single particle, that is to say, we define $\tau(A) := \inf\{t > 0 : \#\{\hat{\xi}_t^x;\ x \in A\} = 1\}$. Take a sequence $\{\varepsilon_N\}$ of positive numbers such that $\varepsilon_N \to 0$ and $N\varepsilon_N \to \infty$ as $N \to \infty$. Moreover, we suppose that when $N \to \infty$,

$$N \cdot P(\xi^0_{\varepsilon_N} = 0) \to 0 \quad \text{and} \quad \sum_{e \in \mathbb{S}_N} p_N(e) \cdot P(\tau(\{0,e\}) \in (\varepsilon_N, t]) \to 0 \quad (\forall t > 0). \tag{3}$$

We also assume now that the following limits exist :

$$\lim_{N\to\infty} \sum_{e \in \mathbb{S}_N} p_N(e) \cdot P(\tau(\{0,e\}) > \varepsilon_N) = \exists\gamma (> 0) \tag{4}$$

$$\text{and} \qquad \lim_{N\to\infty} P\left(\tau(A/\ell_N) \leqslant \varepsilon_N\right) = \exists\zeta(A) \tag{5}$$

holds for any finite subset $A \subset \mathbb{Z}^d$.

Theorem. (cf.[1],[2]) When we denote the law of a measure-valued stochastic process $X_\cdot^N$ on the path space Ω_D by P_N, then there exists a probability measure $P^* \in \mathcal{P}(\Omega_C)$ such that

$$P_N \quad \Longrightarrow \quad P^*_{X_0} \qquad (\text{as} \quad N \to \infty). \tag{6}$$

Then there exists a $M_F(\mathbb{R}^d)$-valued stochastic process $X_t = X_t^{2\gamma,\theta,\sigma^2}$ named a DW superprocess with parameters 2γ, θ and σ^2, satisfying that X_t^N converges to $X_t^{2\gamma,\theta,\sigma^2}$ as $N \to \infty$ in the sense of weak convergence for measures, and $P^*_{X_0}$ is the law of $X_t^{2\gamma,\theta,\sigma^2}$.

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