Lectures on approximate-flows and rough flows

It was realized in the late 70's that stochastic differential equations not only define individual trajectories, they also define flows of regular homeomorphisms, depending on the regularity of the vector fields involved in the dynamics. This opened the door to the study of stochastic flows of maps for themselves, and it did not took long time before Le Jan and Watanabe clarified definitely the situation by showing that, in a semimartingale setting, there is a one-to-one correspondence between flows of diffeomorphisms and time-varying stochastic velocity fields, under proper regularity conditions on the objects involved. We offered in the work [1] an embedding of the theory of semimartingale stochastic flows into the theory of rough flows similar to the embedding of the theory of stochastic differential equations into the theory of rough differential equations.

It is based on the "approximate flow-to-flow" machinery introduced in [2], which gives body to the following fact. To a 2-index family $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$ of maps which falls short from being a flow, in a quantitative way, one can associate a unique flow $(\varphi_{ts})_{0\leqslant s\leqslant t\leqslant T}$ close to $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$; moreover the flow φ depends continuously on the approximate flow μ. The point about such a machinery is that approximate flows appear naturally in a number of situations as simplified descriptions of complex evolutions, often under the form of Taylor-like expansions of complicated dynamics. The model situation is given by a controlled ordinary differential equation

$$\dot{x}_t = \sum_{i=1}^{\ell} V_i(x_t)\,\dot{h}^i_t, \tag{1}$$

in $\mathbb{R}^d$, driven by an $\mathbb{R}^\ell$-valued $\mathcal{C}^1$ control h. The Euler scheme

$$\mu_{ts}(x) = x + \big(h^i_t - h^i_s\big)V_i(x)$$

defines, under proper regularity conditions on the vector fields, an approximate flow whose associated flow is the flow generated by equation (1). One step farther, if we are given a weak geometric Hölder p-rough path $\mathbf{X}$, with $2\leqslant p<3$, and sufficiently regular vector fields $\mathrm{F} = \big(V_1,\dots,V_\ell\big)$ on $\mathbb{R}^d$, one can associate to the rough differential equation

$$dx_t = \mathrm{F}(x_t)\mathbf{X}(dt), \tag{2}$$

some maps μ_{ts} defined, for each $0\leqslant s\leqslant t\leqslant T$, as the time 1 map of an ordinary differential equation involving $\mathbf{X}_{ts}$, the V_i and their brackets, that have the same Taylor expansion as the awaited Taylor expansion of a solution flow to equation (2). They happen to define an approximate flow whose associated flow is the solution flow to equation (2).

A similar approach can be used to deal with a general class of stochastic time-dependent velocity fields. We introduced for that purpose in [1] a notion of rough driver, that is an enriched version of a time-dependent vector field, that will be given by the additional datum of a time-dependent second order differential operator satisfying some algebraic and analytic conditions. A notion of solution to a differential equation driven by a rough driver will be given, in the line of what was done in [2]for rough differential equations, and the approximate flow-to-flow machinery will be seen to lead to a clean and simple well-posedness result for such equations. As awaited from the above discussion, the main point of this result is that the Itô map, that associates to a rough driver the solution flow to its associated equation, is continuous. This continuity result is the key to deep results in the theory of stochastic flows.

We proved indeed in [1] that reasonable semimartingale velocity fields can be lifted to rough drivers under some mild boundedness and regularity conditions, and that the solution flow associated to the 'semimartingale' rough driver coincides almost surely with the solution flow to the Kunita-type Stratonovich differential equation driven by the velocity field. As a consequence of the continuity of the Itô map, a Wong-Zakai theorem was proved for a general class of semimartingale velocity fields, together with sharp support and large deviation theorems for Brownian flows.

These two lectures will introduce the audience to the core of the machinery of approximate and rough flows. The "approximate-flow-to-flow" machinery will be introduced in lecture 1, and used to get back the basics of Lyons' theory of rough differential equations. Lecture 2 will set the scene of rough drivers and rough flows, with hints as to how on can embed the theory of stochastic flows of homeomorphisms into the theory of rough flows.

References

[1] I. Bailleul and S. Riedel, Rough flows, arXiv:1505.01692, (2015).
[2] I. Bailleul, Flows driven by rough paths, *Revista Math. Iberoamericana*, **31**(3) (2015).