Regularization of Generalized Wiener Functionals by Bochner Integral

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Outline

1. Introduction.
2. Regularizing effect under Wiener process.
3. Regularizing effect under diffusion process.

Introduction.

Aim 1

The local time (at zero) of the one-dimensional Wiener process $w = (w(t))_{t \geqslant 0}$ starting from zero is

$$\text{heuristically} \qquad \text{“}\int_0^t \delta_0(w(s))\mathrm{d}s\text{”}, \qquad ①$$

$$\text{rigorously} \qquad \lim_{\varepsilon \to 0} \int_0^t \varphi_\varepsilon(w(s))\mathrm{d}s$$

with a family of functions $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ such that $\varphi_\varepsilon \overset{\varepsilon \to 0}{\to} \delta_0$ in $\mathscr{S}'(\mathbb{R})$.

In this talk, our first aim is **to formulate ① directly as a Bochner integral**.

Known About Donsker's Delta

Donsker's δ-function $\delta_0(w(s))$ has formulations

- ▷ (Kuo '82) as a **generalized Brownian functional** via white noise calculus.
- ▷ (Watanabe '84) as a **generalized Wiener functional**. **We employ this**.

More specifically,

- ▷ (Nuarart-Vives '93 and Watanabe '91 & '94)

$$\delta_0(w(s)) \in \mathbb{D}_2^{(-1/2)-} = \bigcap_{\varepsilon>0} \mathbb{D}_2^{(-1/2)-\varepsilon}$$

Hereafter, $\mathbb{D}_p^s$, $p \in (1,\infty)$, $s \in \mathbb{R}$ denotes the so-called Meyer-Watanabe spaces.

Aim 2: "Regularizing Effect" I

We will prove the Bochner integrability of the mapping

$$(0, t] \ni s \mapsto \delta_0(w(s)) \in \mathbb{D}_2^{-(1/2)-\varepsilon} \quad \text{for } \varepsilon > 0.$$

Note: $\delta_0(w(0))$ no longer makes sense as a generalized Wiener functional, and hence the Bochner integrability is not immediate from the continuity of $s \mapsto \delta_0(w(s))$.

After showing that, the Bochner integral is defined and

$$\int_0^t \delta_0(w(s))\mathrm{d}s \in \mathbb{D}_2^{-(1/2)-\varepsilon} \quad \text{for } \varepsilon > 0.$$

Aim 2: “Regularizing Effect” II

However, the local time is usually defined as a classical Wiener functional, so that it should be in $\mathbb{D}_2^0 = L_2$.

Hence the Bochner integral should pose a sort of “**regularizing effect**”. This phenomenon might be a common understanding at the level of intuition for most of us, but there have not been literatures on this subject except for the case of local time.

Our second aim is **to understand this phenomenon in terms of Meyer-Watanabe spaces**.

Regularizing effect under Wiener process.

Exhibiting the Regularizing Effect I

The following is the prototype of this study.

Theorem

Let $\Lambda \in \mathscr{S}'(\mathbb{R})$, $t > 0$ *and* $s \in \mathbb{R}$. *If* $\Lambda(w(t)) \in \mathbb{D}_2^s$ *then the mapping*

$$(0, t] \ni u \mapsto \sqrt{\frac{t}{u}}\Lambda\big(\sqrt{\frac{t}{u}}w(u)\big) \in \mathbb{D}_2^s$$

is Bochner integrable and

$$\int_0^t \sqrt{\frac{t}{u}}\Lambda\big(\sqrt{\frac{t}{u}}w(u)\big)\mathrm{d}u \in \mathbb{D}_2^{s+1}.$$

Exhibiting the Regularizing Effect II

In particular, we recover that

$$\int_0^t \delta_0(w(s))\mathrm{d}s \in \mathbb{D}_2^{(1/2)-} = \bigcap_{\varepsilon>0} \mathbb{D}_2^{(1/2)-\varepsilon}$$

which agrees with results by Nualart-Vives '92 & Watanabe '94. For higher dimensional local times, we refer Takanobu '04 and Uemura '01, '04.

Exhibiting the Regularizing Effect: Proof. I

Proof. Let H_n: the n-th Hermite polynomial for $n \in \mathbb{Z}_{\geqslant 0}$ such that $\{\frac{1}{\sqrt{n!}} H_n\}_{n=0}^{\infty}$ is a CONB of $L_2(\mathbb{R}, \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}} \mathrm{d}x)$.

Bochner integrability: We have

$$\Lambda\Big(\sqrt{\frac{t}{u}} w(u)\Big) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{E}\Big[\Lambda\Big(\sqrt{\frac{t}{u}} w(u)\Big) H_n\Big(\frac{w(u)}{\sqrt{u}}\Big)\Big] H_n\Big(\frac{w(u)}{\sqrt{u}}\Big),$$

so that

$$\Big\|\sqrt{\frac{t}{u}} \Lambda\Big(\sqrt{\frac{t}{u}} w(u)\Big)\Big\|_{2,s}^2 = \frac{t}{u} \sum_{n=0}^{\infty} \frac{(1+n)^s}{n!} \mathbf{E}\Big[\Lambda\Big(\sqrt{\frac{t}{u}} w(u)\Big) H_n\Big(\frac{w(u)}{\sqrt{u}}\Big)\Big]^2.$$

Exhibiting the Regularizing Effect: Proof. II

Continue computation:

$$
\begin{aligned}
&\|\sqrt{\frac{t}{u}}\Lambda\big(\sqrt{\frac{t}{u}}w(u)\big)\|_{2,s}^2 \\
&= \frac{t}{u}\sum_{n=0}^{\infty}\frac{(1+n)^s}{n!}\underbrace{\mathbf{E}\Big[\Lambda\big(\sqrt{\frac{t}{u}}w(u)\big)H_n\Big(\frac{w(u)}{\sqrt{u}}\Big)\Big]^2}_{=\mathbf{E}[\Lambda(w(t))H_n(\frac{w(t)}{\sqrt{t}})]^2} \\
&= \frac{t}{u}\sum_{n=0}^{\infty}\frac{(1+n)^s}{n!}\mathbf{E}\Big[\Lambda(w(t))H_n\Big(\frac{w(t)}{\sqrt{t}}\Big)\Big]^2 = \frac{t}{u}\|\Lambda(w(t))\|_{2,s}^2.
\end{aligned}
$$

Exhibiting the Regularizing Effect: Proof. III

Hence

$$\int_0^t \|\sqrt{\frac{t}{u}}\Lambda\big(\sqrt{\frac{t}{u}}w(u)\big)\|_{2,s}\mathrm{d}u$$
$$= t^{1/2}\|\Lambda(w(t))\|_{2,s}\int_0^t u^{-1/2}\mathrm{d}u < +\infty, \quad \text{Done.}$$

Second assertion: Next we show

$$\int_0^t \sqrt{\frac{t}{u}}\Lambda\big(\sqrt{\frac{t}{u}}w(u)\big)\mathrm{d}u \in \mathbb{D}_2^{s+1}.$$

Exhibiting the Regularizing Effect: Proof. IV

Note that

$$\int_0^t \sqrt{\frac{t}{u}}\Lambda\big(\sqrt{\frac{t}{u}}w(u)\big)\mathrm{d}u$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{E}\big[\Lambda(w(t))H_n\Big(\frac{w(t)}{\sqrt{t}}\Big)\big] \underbrace{\int_0^t \sqrt{\frac{t}{u}}H_n\Big(\frac{w(u)}{\sqrt{u}}\Big)\mathrm{d}u}_{\text{still remains to be an } n\text{-th order chaos}}$$

is an Itô-Wiener expansion of $\int_0^t \sqrt{\frac{t}{u}}\Lambda\big(\sqrt{\frac{t}{u}}w(u)\big)\mathrm{d}u$.

Exhibiting the Regularizing Effect: Proof. V

Therefore

$$
\begin{aligned}
&\Big\| \int_0^t \sqrt{\frac{t}{u}}\Lambda\Big(\sqrt{\frac{t}{u}}w(u)\Big)\mathrm{d}u\Big\|_{2,s+1}^2 \\
&= \sum_{n=0}^{\infty} \frac{(n+1)^{s+1}}{(n!)^2} \mathbf{E}\Big[\Lambda(w(t))H_n\Big(\frac{w(t)}{\sqrt{t}}\Big)\Big]^2 \\
&\qquad\qquad \times \underbrace{\mathbf{E}\Big[\Big\{\int_0^t \sqrt{\frac{t}{u}}H_n\Big(\frac{w(u)}{\sqrt{u}}\Big)\mathrm{d}u\Big\}^2\Big]}_{\text{need to compute}}.
\end{aligned}
$$

Exhibiting the Regularizing Effect: Proof. VI

$$\mathbf{E}\Big[\Big\{\int_0^t \sqrt{\frac{t}{u}}H_n\Big(\frac{w(u)}{\sqrt{u}}\Big)\mathrm{d}u\Big\}^2\Big]$$

$$= 2t\int_{0<u<v<t}\frac{1}{\sqrt{uv}}\mathbf{E}\Big[H_n\Big(\frac{w(u)}{\sqrt{u}}\Big)H_n\Big(\underbrace{\frac{w(v)}{\sqrt{v}}}_{\sqrt{\frac{u}{v}}\frac{w(u)}{\sqrt{u}}+\frac{w(v)-w(u)}{\sqrt{v}}}\Big)\Big]\mathrm{d}u\mathrm{d}v$$

$$= 2t\,n!\int_{0<u<v<t}\frac{1}{\sqrt{uv}}\Big(\frac{u}{v}\Big)^{n/2}\mathrm{d}u\mathrm{d}v = \frac{4t^2}{n+1}n!.$$

Exhibiting the Regularizing Effect: Proof. VII

Hence

$$
\begin{aligned}
&\| \int_0^t \sqrt{\frac{t}{u}} \Lambda\big(\sqrt{\frac{t}{u}} w(u)\big) \mathrm{d}u \|_{2,s+1}^2 \\
&= \sum_{n=0}^{\infty} \frac{(n+1)^{s+1}}{(n!)^2} \mathbf{E}\Big[\Lambda(w(t)) H_n\Big(\frac{w(t)}{\sqrt{t}}\Big)\Big]^2 \frac{4t^2}{n+1} n! \\
&= 4t^2 \sum_{n=0}^{\infty} \frac{(n+1)^s}{n!} \mathbf{E}\Big[\Lambda(w(t)) H_n\Big(\frac{w(t)}{\sqrt{t}}\Big)\Big]^2 \\
&= 4t^2 \|\Lambda(w(t))\|_{2,s}^2 < +\infty. \qquad \square
\end{aligned}
$$

Regularizing effect under diffusion process.

In the sequel, we shall consider a "regularizing effect" for the local time "$\int_0^t \delta_y(X(s,x,w))\mathrm{d}s$" in a much weaker form, where $\{X(t,x,w)\}_{t\geqslant 0}$ is a unique strong solution to 1-dim. SDE

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}w(t) + b(X_t)\mathrm{d}t$$

such that $X(0,x,w) = x \in \mathbb{R}$.

Assumption.

(i) $\sigma, b : \mathbb{R} \to \mathbb{R}$ are C^∞ and bounded,

(ii) all derivatives of σ and b are bounded,

(iii) $\inf_{x\in\mathbb{R}} \sigma(x)^2 > 0$.

Bochner Integrability Under Diffusion

A distribution $\Lambda \in \mathscr{S}'(\mathbb{R})$ is said to be **positive** if

$$\langle \Lambda, f \rangle \geqslant 0 \quad \forall \text{nonnegative } f \in \mathscr{S}(\mathbb{R}).$$

This condition is known to imply

- $\exists$ a non-negative Radon measure μ on $\mathbb{R}$ s.t.

$$\langle \Lambda, f \rangle = \int_{\mathbb{R}} \langle \delta_y, f \rangle \mu(\mathrm{d}y) \quad \forall f \in \mathscr{S}(\mathbb{R}).$$

Theorem

Let $x \in \mathbb{R}$, $t > 0$ and $\Lambda \in \mathscr{S}'(\mathbb{R})$ be positive. Then

$$\forall p \in (1, \infty), \quad \int_0^t \int_{\mathbb{R}} \|\delta_y(X(s, x, w))\|_{p,-2} \mu(\mathrm{d}y) \mathrm{d}s < +\infty,$$

where μ is the Radon measure associated to Λ.

Sketch of Proof

Take $K > 0$ s.t.

$$|x-y|^2 \geqslant \frac{y^2}{2} \quad \text{for any } |y| > K$$

and then we divide the integral

$$\int_0^t \int_{\mathbb{R}} \|\delta_y(X(s,x,w))\|_{p,-2}\mu(\mathrm{d}y)\mathrm{d}s$$
$$= \underbrace{\int_0^t \int_{|y|>K} \|\delta_y(X(s,x,w))\|_{p,-2}\mu(\mathrm{d}y)\mathrm{d}s}_{=:\, I_1}$$
$$+ \underbrace{\int_0^t \int_{|y|\leqslant K} \|\delta_y(X(s,x,w))\|_{p,-2}\mu(\mathrm{d}y)\mathrm{d}s}_{=:\, I_2}.$$

Sketch of Proof: Estimate of I_1 I

Proposition

There exist $\nu_0, c_1, c_2 > 0$ *s.t.*

$$\|\delta_y(X(s,x,w))\|_{p,-2} \leqslant c_1 s^{-\nu_0} \exp\left\{ - c_2 \frac{|x-y|^2}{s} \right\}$$

for any $s \in (0,t]$ *and* $y \in \mathbb{R}$ *with* $|y| \geqslant K$.

Hence we have

$$\begin{aligned} I_1 &= \int_0^t \int_{|y|>K} \|\delta_y(X(s,x,w))\|_{p,-2} \mu(\mathrm{d}y)\mathrm{d}s \\ &\leqslant c_1 \int_0^t \int_{|y|>K} s^{-\nu_0} \exp\Big\{ - c_2 \underbrace{\frac{|x-y|^2}{s}}_{\geqslant \frac{|y|^2}{2s}} \Big\} \mu(\mathrm{d}y)\mathrm{d}s \end{aligned}$$

Sketch of Proof: Estimate of I_1 II

$$\leqslant c_1 \int_0^t \int_{|y|>K} \underbrace{s^{-\nu_0} \exp\left\{ -c_2 \frac{|y|^2}{2s} \right\}}_{\leqslant \text{ const. } e^{-c_2 \frac{|y|^2}{4s}}} \mu(\mathrm{d}y)\mathrm{d}s$$

$$\leqslant \text{const.} c_1 \int_0^t \int_{|y|>K} \exp\left\{ -c_2 \underbrace{\frac{|y|^2}{4s}}_{\geqslant \frac{|y|^2}{4t}} \right\} \mu(\mathrm{d}y)\mathrm{d}s$$

$$\leqslant \text{const.} c_1\, t \int_{\mathbb{R}} \underbrace{\exp\left\{ -c_2 \frac{|y|^2}{4t} \right\}}_{=:\, \varphi(y)} \mu(\mathrm{d}y)$$

$$\leqslant \text{const.} c_1\, t \langle \Lambda, \varphi \rangle < +\infty \quad \text{because } \varphi \in \mathscr{S}(\mathbb{R}).$$

Sketch of Proof: Estimate of I_2 I

Consider $\{X^\varepsilon(t,x,w)\}_{t\geqslant 0}$, the strong solution to

$$\mathrm{d}X_t = \varepsilon\sigma(X_t)\mathrm{d}w(t) + \varepsilon^2 b(X_t)\mathrm{d}t.$$

Well known: $\{X(\varepsilon^2 t,x,w)\}_{t\geqslant 0} \overset{\text{in law}}{=} \{X^\varepsilon(t,x,w)\}_{t\geqslant 0}$ $\forall \varepsilon > 0$.

A more tricky fact which we need is

Lemma

$\forall p \in (1,\infty)$, $\forall s \in \mathbb{R}$ and $\forall T \in \mathscr{S}'(\mathbb{R})$,

$$\|T(X(\varepsilon^2 t,x,w))\|_{p,s} = \|T(X^\varepsilon(t,x,w))\|_{p,s}.$$

Sketch of Proof: Estimate of I_2 II

Therefore we have

$$
\begin{aligned}
& s^{1/2} \sup_{y\in\mathbb{R}} \|\delta_y(X(s,x,w))\|_{p,-2} \\
&= s^{1/2} \sup_{y\in\mathbb{R}} \|\delta_y(X^{\sqrt{s}}(1,x,w))\|_{p,-2} \\
&= \sup_{y\in\mathbb{R}} \|\delta_{(x-y)/\sqrt{s}}\Big(\frac{X^{\sqrt{s}}(1,x,w)-x}{\sqrt{s}}\Big)\|_{p,-2} \\
&\leqslant \text{const.} \sup_{a\in\mathbb{R}} \|\delta_a\|_{-2} < +\infty,
\end{aligned}
$$

where $\|\delta_a\|_{-2} := \|(1+z^2-\frac{\mathrm{d}^2}{\mathrm{d}z^2})^{-1}\delta_a\|_\infty$.

Sketch of Proof: Estimate of I_2 III

Hence we obtain

$$\begin{aligned} I_2 &= \int_0^t \int_{|y|\leqslant K} \|\delta_y(X(s,x,w))\|_{p,-2}\mu(\mathrm{d}y)\mathrm{d}s \\ &\leqslant \text{const.}\ \underbrace{\mu\big(\{y \in \mathbb{R} : |y| \leqslant K\}\big)}_{< +\infty \text{ since } \mu \text{ is Radon.}} \int_0^t s^{-1/2}\mathrm{d}s < +\infty. \qquad \square \end{aligned}$$

A Generalized Itô Formula I

Let $\Lambda \in \mathscr{S}'(\mathbb{R})$ and $k \in \mathbb{N}$. If $\int_0^t \|\Lambda(X(s,x,w))\|_{2,-k}^2 \mathrm{d}s < +\infty$, we define the **stochastic integral** $\int_0^t \Lambda(X(s,x,w))\mathrm{d}w(s) \in \mathbb{D}^{-\infty}$ by the pairing

$$\mathbf{E}\Big[\Big\{\int_0^t \Lambda(X(s,x,w))\mathrm{d}w(s)\Big\}J\Big] = \int_0^t \mathbf{E}[\Lambda(X(s,x,w))D_sJ]\mathrm{d}s$$

for $J \in \mathbb{D}^\infty$. This is well defined by

Proposition (cf. Uemura '04)

For $J \in \mathbb{D}^\infty$, we have

$$\begin{aligned}&\int_0^t |\mathbf{E}[\Lambda(X(s,x,w))D_sJ]|\mathrm{d}s \\ &\leqslant \mathrm{const.}\Big\{\int_0^t \|\Lambda(X(s,x,w))\|_{2,-k}^2 \mathrm{d}s\Big\}^{1/2} \|J\|_{2,k+1}.\end{aligned}$$

A Generalized Itô Formula II

Let $V := \sigma \frac{\mathrm{d}}{\mathrm{d}z}$ and $L := \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + b\frac{\mathrm{d}}{\mathrm{d}z}$.

Theorem (cf. Kubo '83)

Let $f : \mathbb{R} \to \mathbb{R}$ be a m'ble function such that

(i) *f is continuous at x,* (ii) *$f \in \mathscr{S}'(\mathbb{R})$,*

(iii) $\displaystyle\int_0^t \|(Vf)(X(s,x,w))\|_{2,-k}^2 \mathrm{d}s < +\infty$,

(iv) $\displaystyle\int_0^t \|(Lf)(X(s,x,w))\|_{2,-k} \mathrm{d}s < +\infty$

for some $k \in \mathbb{N}$. Then we have

$$
\begin{aligned}
&f(X(t,x,w)) - f(x) \\
&= \int_0^t (Vf)(X(s,x,w))\mathrm{d}w(s) + \int_0^t (Lf)(X(s,x,w))\mathrm{d}s \quad \textit{in } \mathbb{D}^{-\infty}.
\end{aligned}
$$

Regularizing Effect in Weaker Form I

We fix $y \in \mathbb{R}$ and set for $x \in \mathbb{R}$,

$$s(x) := \int_0^x \exp\Big\{ - \int_y^z \frac{2b(\eta)}{\sigma(\eta)^2} \mathrm{d}\eta \Big\} \mathrm{d}z,$$
$$m(x) := 2\int_0^x \exp\Big\{ \int_y^z \frac{2b(\eta)}{\sigma(\eta)^2} \mathrm{d}\eta \Big\} \frac{\mathrm{d}z}{\sigma(z)^2}.$$

Define $u : \mathbb{R} \to \mathbb{R}$ by

$$u(x) := \frac{m'(y)}{2} |s(x) - s(y)|, \quad x \in \mathbb{R}.$$

Assumption.

(iv) $\exists c > 0$ & $p > 1$ s.t.

$$\Big| \int_y^z \frac{2b(\eta)}{\sigma(\eta)^2} \mathrm{d}\eta \Big| \leqslant c \log(1 + |z-y|^p), \quad z \in \mathbb{R}.$$

($\rightsquigarrow u \in \mathscr{S}'(\mathbb{R})$.)

Regularizing Effect in Weaker Form II

Lemma

We have $Lu = \delta_y$ in the distributional sense.

Hence we have

$$\begin{aligned} & u(X(t,x,w)) - u(x) \\ & = \int_0^t (Vu)(X(s,x,w))\mathrm{d}w(s) + \int_0^t \delta_y(X(s,x,w))\mathrm{d}s \end{aligned}$$

in $\mathbb{D}^{-\infty}$, from which we conclude

Corollary

Under (i)-(iv), we have $\int_0^t \delta_y(X(s,x,w))\mathrm{d}s \in L_2$.

Thank you for your attention.