Regularization of Generalized Wiener Functionals by Bochner Integral

Takafumi Amaba[♭1]

[1]Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan

The local time (at zero) of the one-dimensional Wiener process $w = (w(t))_{t\geq 0}$ starting from zero is heuristically written as

$$\text{“}\int_0^t \delta_0(w(s))\mathrm{d}s\text{”} \tag{1}$$

and is rigorously formulated by

$$\lim_{\varepsilon\to 0}\int_0^t \varphi_\varepsilon(w(s))\mathrm{d}s$$

with using rapidly decreasing functions $\{\varphi_\varepsilon\}_{\varepsilon>0}$ which tends to Dirac's delta function δ_0 in the space of Schwartz distributions.

In this talk, we formulate (1) directly as a Bochner integral. It is well known that $\delta_0(w(t))$ makes sense as a generalized Wiener functional (see Watanabe [4]) and belongs to $\mathbb{D}_2^{(-1/2)-\varepsilon}$ for each $\varepsilon > 0$. (see Nualart-Vives [2], Watanabe [5, 6]). For our objective, we need to consider the Bochner integrability of the mapping

$$(0,t] \ni s \mapsto \delta_0(w(s)) \in \mathbb{D}_2^{(-1/2)-\varepsilon}. \tag{2}$$

Note that $\delta_0(w(0))$ does not make sense as a generalized Wiener functional, and hence the Bochner integrability does not follow only from the continuity of the mapping $t \mapsto \delta_0(w(t))$.

When succeeded in seeing the Bochner integrability of the mapping (2), the Bochner integral $\int_0^t \delta_0(w(s))\mathrm{d}s$ makes sense as an element in $\mathbb{D}_2^{-(1/2)-\varepsilon}$ for $\varepsilon > 0$. However, the local time is known to be a classical Wiener functional, so that it should be in $\mathbb{D}_2^0 = L_2$. Hence, the Bochner integral should pose a sort of “regularizing effect”. This phenomenon might be a common understanding at the level of intuition for most of us, but there have not been literatures on this subject except for the case of local times.

Denote by $\mathscr{S}'(\mathbb{R})$ the space of Schwartz distributions on $\mathbb{R}$. The following is the prototype of this study:

Theorem 1. *Let $\Lambda \in \mathscr{S}'(\mathbb{R})$, $t > 0$ and $s \in \mathbb{R}$. If $\Lambda(w(t)) \in \mathbb{D}_2^s$ then the mapping*

$$(0,t] \ni u \mapsto \sqrt{\frac{t}{u}}\Lambda\Big(\sqrt{\frac{t}{u}}w(u)\Big) \in \mathbb{D}_2^s$$

[♭]The author was supported by JSPS KAKENHI Grant Number 15K17562.

fm-amaba@fc.ritsumei.ac.jp

is Bochner integrable in $\mathbb{D}_2^s$ *and we have*

$$\int_0^t \sqrt{\frac{t}{u}}\Lambda\Big(\sqrt{\frac{t}{u}}w(u)\Big)\mathrm{d}u \in \mathbb{D}_2^{s+1}.$$

From this we obtain $\int_0^t \delta_0(w(u))\mathrm{d}u \in \mathbb{D}_2^{(1/2)-\varepsilon}$ for each $\varepsilon > 0$ which agrees with the results in [2, 3, 6].

The proof of the above theorem is obtained by looking at all chaos appearing in the Itô-Wiener expansion for $\Lambda((t/u)^{1/2}w(u))$ and relies on their explicit forms, so that it is hard to obtain a similar result for diffusion processes.

We will consider a similar "regularizing effect" for $\int_0^t \delta_0(X(s,x,w))\mathrm{d}s$ from purely Malliavin calculus-viewpoint, where $X(t,x,w)$ is the unique strong solution to a one-dimensional stochastic differential equation

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}w(t) + b(X_t)\mathrm{d}t.$$

Although we have not succeed to obtain a similar result to above, we can show the Bochner integrability of the mapping $u \mapsto \delta_0(X(u,x,w))$ and that $\int_0^t \delta_0(X(u,x,w))\mathrm{d}u \in L_2$. The proof is based on Itô's formula for generalized Wiener functionals which is a slight extension of Kubo [1].

References

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