2015年度　確率解析とその周辺

(Stochastic Analysis and Related Topics 2015)

予稿集

(Abstracts)

2015年 10月20日(火) 13:40 ～ 10月22日(木) 16:00

(Oct. 20 (Tue.) – Oct. 22 (Thu.), 2015)

大阪大学大学院基礎工学研究科
基礎工学国際棟 (Σホール) セミナー室

(International Hall, Graduate School of Engineering Science, Osaka University)

研究集会「確率解析とその周辺」

平成 27 年度科学研究費補助金基盤研究 (B) 課題番号 24340023「無限次元空間上の確率解析」(研究代表者：会田 茂樹), 平成 27 年度科学研究費補助金基盤研究 (B) 課題番号 15H03624「確率解析的手法によるマルコフ過程の研究と応用」(研究代表者：重川 一郎) の援助を受けて, 表記の研究集会を以下の要領で開催致しますのでご案内申し上げます。

日時： 2015 年 10 月 20 日（火）13:40 〜 10 月 22 日（木）16:00

場所： 大阪大学大学院基礎工学研究科 基礎工学国際棟 (Σ ホール) セミナー室
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—プログラム—

10 月 20 日 (火)

13:40〜14:20　植村 英明 (Hideaki Uemura)　(愛知教育大学)
On the derivation of noncausal function from its Haar-SFCs

14:30〜15:30　福島 正俊 (Masatoshi Fukushima)　(大阪大学)　[特別講演]
Locality property and a related continuity problem for SLE and SKLE (1)

15:50〜16:30　吉川 和宏 (Kazuhiro Yoshikawa)　(立命館大学)
SLK martingales and representations of the Witt algebra

16:40〜17:20　道工 勇 (Isamu Dôku)　(埼玉大学)
On convergence of environment-dependent models

10 月 21 日 (水)

9:30〜10:10　深澤 正彰 (Masaaki Fukasawa)　(大阪大学)
Phase transitions in a control problem

10:20〜11:00　楠岡 誠一郎 (Seiichiro Kusuoka)　(岡山大学)
The rates of the L^p-convergence of the Euler–Maruyama and the Wong–Zakai approximations of path-dependent stochastic differential equations under the Lipschitz condition

11:10〜12:10　Ismaël Bailleul　(Université de Rennes 1)　[特別講演]
Approximate-flows and rough flows (1)

12:10〜13:40　昼休み

13:40〜14:20　鈴木 康平 (Kohei Suzuki)　(京都大学)
Convergence of Brownian motions on RCD spaces

14:30〜15:30　福島 正俊 (Masatoshi Fukushima)　(大阪大学)　[特別講演]
Locality property and a related continuity problem for SLE and SKLE (2)

15:50〜16:30　星野 壮登 (Masato Hoshino)　(東京大学)
KPZ equation with fractional derivatives of white noise

16:40〜17:20　Ta Viet Ton　(大阪大学)
Existence and uniqueness of strict solutions of stochastic linear evolution equations in M-type 2 Banach spaces

10 月 22 日 (木)

9:30〜10:10　中津 智則 (Tomonori Nakatsu)　(立命館大学)
On density function concerning discrete time maximum of some one-dimensional diffusion processes

10:20〜11:00　永沼 伸顕 (Nobuaki Naganuma)　(東北大学)
Error analysis for approximations to one-dimensional SDEs via perturbation method

11:10〜12:10　Ismaël Bailleul　(Université de Rennes 1)　[特別講演]
Approximate-flows and rough flows (2)

12:10〜13:40　昼休み

13:40〜14:20　田口 大 (Dai Taguchi)　(立命館大学)
Parametrix method for skew diffusion

14:30〜15:10　天羽 隆史 (Takafumi Amaba)　(立命館大学)
Regularization of generalized Wiener functionals by Bochner integral

15:20〜16:00　稲浜 譲 (Yuzuru Inahama)　(九州大学)
Large deviations for rough path lifts of Donsker–Watanabe's delta functions

世話人　重川 一郎（京都大学大学院理学研究科）
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On the derivation of noncausal function from its Haar-SFCs *

Shigeyoshi OGAWA (Ritsumeikan University)
Hideaki UEMURA (Aichi University of Education)

(i) SFC. Let $f(t,\omega)$ be a random function on $[0,1]\times\Omega$ and $\{\varphi_n(t)\}$ be a CONS in $L^2([0,1];\mathbb{C})$. The system $\{\hat{f}_n(\omega)=\int_0^1 f(t,\omega)\overline{\varphi_n(t)}dW_t\}$ is called the *stochastic Fourier coefficients* (*SFCs* in abbr.) of $f(t,\omega)$, $\{W(t), t\in[0,1]\}$ being a Brownian motion on $(\Omega,\mathcal{F},P)$ which starts at the origin. It is of course that the stochastic integral $\int dW$ in the definition of SFCs should adequately be chosen according to the conditions on $f(t,\omega)$. We are concerned with the problem whether $f(t,\omega)$ is identified from the SFCs of $f(t,\omega)$ or not.

In this talk we consider the case that $f(t,\omega)$ is noncausal, and we aimed to identify $f(t,\omega)$ without the aid of a Brownian motion. Moreover, we intend to derive $f(t,\omega_0)$ from SFCs $\{\hat{f}_n(\omega_0)\}$ for almost all ω_0.

(ii) SFT. Let $\{\varepsilon_n\}$ be an ℓ_2 sequence such that $\varepsilon_n\neq 0$ for all n. Then

$$\mathcal{T}_{(\varepsilon,\varphi)}(f)(t,\omega)=\sum_n \varepsilon_n\hat{f}_n(\omega)\varphi_n(t)$$

is called $(\varepsilon_n,\varphi_n)$-*stochastic Fourier transform* (*SFT* in abbr.) of $f(t,\omega)$. In [1] we discussed this problem under the condition that SFCs are defined by employing the Ogawa integral as a stochastic integral and the system of trigonometric functions $e_n(t)=\mathrm{e}^{2\pi int}$, $n\in\mathbb{Z}$, as a CONS. We assumed the next three conditions on $f(t,\omega)$;

[H1] For almost all ω, $f(t,\omega)$ is a differentiable function with respect to t satisfying $f'(t,\omega)\in L^2([0,1],dt)$, where $f'(t,\omega)=\partial f(t,\omega)/\partial t$.

[H2] $\int_0^1 f(t,\omega)dt\in L^2(\Omega,dP)$ and $f'(t,\omega)\in L^2([0,1]\times\Omega,dtdP)$.

[H3] For almost all ω, $f(t,\omega)$ is a nonnegative function.

[H.1] assures us of the existence of SFCs, and the (τ_n,e_n)-SFT $\mathcal{T}_{(\tau,e)}(f)(t,\omega)$ of $f(t,\omega)$ exists in $C^1(0,1)$ under the condition [H.2], where $\tau_n=(-4\pi^2n^2)^{-1}$ if $n\neq 0$ and $\tau_0=1$. From [H.3] and the law of iterated logarithm of the Brownian motion we have

$$P\left(\limsup_{h\downarrow 0}\frac{\mathcal{T}_{(\tau,e)}(f)'(t+h,\omega)-\mathcal{T}_{(\tau,e)}(f)'(t,\omega)}{\sqrt{2h\log\log\frac{1}{h}}}=f(t,\omega),\quad \forall t\in\mathbb{T}\right)=1,$$

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where $\mathbb{T}$ is an arbitrary dense subset of $(0,1)$.

(iii) Haar-SFC. In this talk we employ the Ogawa integral and the system of Haar functions to define SFCs of $f(t,\omega)$. We assume the next two conditions on $f(t,\omega)$;

[H1'] For almost all ω, $f(t,\omega)$ is a continuous function on $[0,1]$ satisfying there exists a function $g(s,\omega) \in L^2([0,1], ds)$ such that $f(t,\omega) - f(0,\omega) = \int_0^t g(s,\omega)ds$.

[H3] For almost all ω, $f(t,\omega)$ is a nonnegative function.

Let $\{H_k^{(n)}; (n,k) \in \Lambda\}$ be the system of Haar functions on $[0,1]$, i.e., $H_0^{(0)}(t) = 1$ and

$$H_k^{(n)}(t) = \begin{cases} 2^{(n-1)/2} & (t_{n.2k} \le t < t_{n.2k+1}) \\ -2^{(n-1)/2} & (t_{n.2k+1} \le t < t_{n.2k+2}) \\ 0 & (\text{otherwise}) \end{cases}$$
$$(n = 1, 2, \dots,\ k = 0, 1, \dots, 2^{n-1} - 1),$$

where $t_{n,k} = k/2^n$. We denote the Haar SFC corresponding to $H_k^{(n)}(t)$ by $\hat{f}_k^{(n)}(\omega)$:

$$\hat{f}_k^{(n)}(\omega) = \int_0^1 f(t,\omega) H_k^{(n)}(t) d_* W_t,$$

$\int d_* W_t$ denoting the Ogawa integral. Set

$$S_N(t.\omega) = \hat{f}_0^{(0)}(\omega) H_0^{(0)}(t) + \sum_{n=1}^{N} \sum_{k=0}^{2^{n-1}-1} \hat{f}_k^{(n)}(\omega) H_k^{(n)}(t).$$

Then we have the following lemma;

Lemma 1. *For $t \in [t_{N,\ell}, t_{N,\ell+1})$, $\ell = 0, 1, \dots, 2^N - 1$, it holds that*

$$S_N(t.\omega) = 2^N \left[f(t_{N,\ell+1},\omega) W(t_{N,\ell+1}) - f(t_{N,\ell},\omega) W(t_{N,\ell}) - \int_{t_{N,\ell}}^{t_{N,\ell+1}} g(t,\omega) W(t) dt \right].$$

From [H.3] and the law of iterated logarithm of the Brownian motion we have our main theorem;

Theorem 1. *Suppose that $f(t,\omega)$ satisfies conditions [H.1'] and [H.3]. Let $\mathbb{T}$ be a countable dense subset of $[0,1)$. Then we have*

$$P\left(\limsup_{N\to\infty} \frac{S_N(t,\omega)}{\sqrt{2^{N+1}\log N}} = f(t,\omega), \quad \forall t \in \mathbb{T} \right) = 1.$$

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Locality property and a related continuity problem for SLE and SKLE

Masatoshi Fukushima(Osaka)
a joint work with Zhen-Qing Chen(Seattle)

The *Schramm Loewner evolution* SLE_κ for $\kappa > 0$ is a family of random growing hulls $\{F_t\}$ in the upper half-plane $\mathbb{H}$ driven by $\xi(t) = B(\kappa t)$ through the *Loewner differential equation*, where $B(t),\ t \geq 0$, is the standard Brownian motion on the boundary $\partial\mathbb{H}$. Early in the 2000s, G. Lawler, G. Schramm and W. Werner observed that SLE_6 enjoys the *locality property* in the sense that, if $\mathbb{H}$ is perturbed away from the hulls, the family of perturbed hulls $\{\widetilde{F}_t\}$ under a due reparametrization has the same distribution as the unperturbed one $\{F_t\}$ (see Figure 1 for percolation). This property can be shown by comparing the driving process $\widetilde{\xi}(t)$ of $\{\widetilde{F}_t\}$ with $B(6t)$ in principle.

But, in doing so rigorously, one need to verify the joint continuity in $(t, \widetilde{z})$ for the family of conformal maps $\widetilde{g}_t(\widetilde{z})$ associated with $\{\widetilde{F}_t\}$, that seems to be left unconfirmed. In this lectures, we will characterize the locality property of stochastic Komatu-Loewner evolutions for multiply connected domains by establishing the stated continuity in this generality.

A *standard slit domain* is a domain of the type $D = \mathbb{H} \setminus \bigcup_{j=1}^N C_j$ for mutually disjoint line segments $C_j \subset \mathbb{H}$ parallel to $\partial\mathbb{H}$. Consider a set $S \subset \mathbb{R}^{3N}$ representing the collection of all labelled slits. Let $(\xi(t), \mathbf{s}(t)) \in \partial\mathbb{H} \times S$ be the strong solution of an SDE such that a diffusion coefficient α and a drift coefficient b of $\xi(t)$ are homogeneous functions of degree 0 and -1, respectively, both satisfying a local Lipschitz continuity condition, while each component of $\mathbf{s}(t)$ has only a drift coefficient determined by the trace to the slits of the *BMD complex Poisson kernel* that is known to be locally Lipschitz continuous.

A *stochastic Komatu-Loewner evolution* denoted by $\mathrm{SKLE}_{\alpha,b}$ is a family of random growing hulls $\{F_t\}$ in a standard slit domain D driven by $(\xi(t), \mathbf{s}(t))$ through the *Komatu-Loewner differential equation*. Let b_{BMD} be the *BMD-domain constant* that describes the discrepancy of a standard slit domain from $\mathbb{H}$ relative to BMD (*Brownian motion with darning*).

Theorem 0.1 $\mathrm{SKLE}_{\alpha,-b_{\mathrm{BMD}}}$ *for a positive constant* α *enjoys the locality property if and only if* $\alpha = \sqrt{6}$.

We use a probabilistic expression of $\Im h_t(z)$ in terms of the BMD and the absorbing Brownian motion (ABM) on D_t and combine it with the conformal invariance of BMD and ABM to obtain an expression of $\Im\widetilde{g}_t(\widetilde{z})$ in terms of $g_t(z)$ (see Figure 3). Note that $g_t(z)$ is jointly continuous as the solution of ODE (KL-equation). This is the way to prove the desired joint continuity of $\widetilde{g}_t(\widetilde{z})$, which additionally yields the joint continuity of $h_t(z), h_t'(z), h_t''(z)$. The last property is crucial to legitimate a use of a generalized Itô formula on a composite of a semi-martingale and a random smooth function formulated by Revuz-Yor in getting an explicit semi-martingale expression of the driving process $\widetilde{\xi}(t)$ of the image hulls $\{\widetilde{F}_t\}$.

Some open problems related to SKLE will be also discussed.

SLK martingales and representations of the Witt algebra

Kazuhiro Yoshikawa (Ritsumeikan University)
Takafumi Amaba * (Ritsumeikan University)

1. Introduction

The Loewner differential equation whose driving function is a Brownian motion is called the Schramm Loewner evolution (SLE). The random coefficients of the expansion of the SLE have a hierarchy of stochastic differential equations, which induces a class of polynomials characterized by martingales. It is known that those polynomials connect the SLE to representations of the Virasoro algebra ([3]). In this talk, we introduce random coefficients based on the Loewner-Kufarev equation and martingales related with the Kirillov-Neretin polynomials. Our aim is to find some relations between stochastic differential equations and representations of the Virasoro algebra

2. A hierarchical solution of stochastic differential equations

Put $D = \{z \in \mathbb{C} \mid |z| < 1\}$. We consider stochastic processes $C(t), c_1(t), c_2(t), \ldots$ generated by a holomorphic function $g_t(z)$ on D :

$$g_t(z) = C(t)(z + c_1(t)z^2 + c_2(t)z^3 + \cdots)$$

which is a solution of the following stochastic differential equation

$$\begin{cases} dg_t(z) = zg_t'(z)\Big\{dX_t^0 + \sum_{k=1}^{\infty} z^k dX_t^k\Big\}, \\ g_0(z) = z \in D. \end{cases} \tag{2.1}$$

Here $X_t^0 = \alpha_0^{-1}t$, $\alpha_0 > 0$, $X_t^k = \alpha_k^{-1}Z_t^k$, $\alpha_k > 0$ for $k \geq 1$ and $Z_t^1, Z_t^2, \ldots$ are infinitely many independent complex Brownian motions. We note that the solution of (2.1) means a hierarchy of stochastic differential equations for the coefficients of the expansion of $g_t(z)$:

$$\begin{cases} dC(t) &= C(t)dX_t^0, \\ dc_1(t) &= dX_t^1 + c_1(t)dX_t^0, \\ dc_2(t) &= dX_t^2 + 2c_1(t)dX_t^1 + 2c_2(t)dX_t^0, \\ dc_3(t) &= dX_t^3 + 2c_1(t)dX_t^2 + 3c_2(t)dX_t^1 + 3c_3(t)dX_t^0, \\ & \vdots \end{cases}$$

In [2], we regarded this hierarchy as a solution of a stochastic Lowener Kufarev equation, which is an approach for constructions of measures on loops.

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3. The Kirillov-Neretin polynomials

We denote by $\mathcal{M}$ the set of all holomorphic functions $f : \bar{D} \to \mathbb{C}$ written as

$$f(z) = z(1 + \sum_{n=1}^{\infty} c_n z^n) \quad \text{for all } z \in D.$$

The Kirillov-Neretin polynomials can be defined as follows:

$$\sum_{n=0}^{\infty} P_n(c_1, \dots, c_n) z^n = h\Big(\frac{zf'(z)}{f(z)}\Big)^2 + \frac{cz^2}{12} S(f)(z), \quad \text{for all } f \in \mathcal{M},$$

where $S(f)$ is the Schwarzian derivative of f :

$$S(f)(z) := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2.$$

In particular, if $h = 0$, then we have

$$P_0 = P_1(c_1) = 0, \quad P_2(c_1, c_2) = \gamma_2(c_2 - c_1^2), \quad P_3(c_1, c_2, c_3) = \gamma_3(c_3 - 2c_1c_2 + c_1^3),$$
$$P_4(c_1, c_2, c_3, c_4) = \gamma_4\big(c_4 - 2c_1c_3 - \frac{6}{5}c_2^2 + \frac{17}{5}c_1^2c_2 - \frac{6}{5}c_1^4\big), \dots,$$

where $\gamma_k := \frac{c}{12}(k^3 - k)$. We can find the formula for Virasoro algebra to act the Kirillov-Neretin polynomials in [1], [4].

4. Martingales based on the Kirillov-Neretin polynomials

Now we put

$$P_n(t) := P_n(c_1(t), \dots, c_n(t)), \quad n \in \mathbb{Z}_{\geq 0}.$$

The first few terms of the stochastic processes $P_n(t)$ are as follows:

$$\begin{aligned}
dP_2(t) &= \gamma_2 dX_t^2 + 2P_2(t)dX_t^0, \\
dP_3(t) &= \gamma_3 dX_t^3 + 4P_2(t)dX_t^1 + 3P_3(t)dX_t^0, \\
dP_4(t) &= \gamma_4 dX_t^4 + 6P_2(t)dX_t^2 + 5P_3(t)dX_t^1 + 4P_4(t)dX_t^0.
\end{aligned}$$

Then, we gain the following result.

Theorem 4.1. *For all* $n = 0, 1, 2, \dots,$

$$e^{-nt/\alpha_0} P_n(t) \text{ is a (local) martingale.}$$

Moreover, these martingales are generated by successive actions of the Witt algebra.

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On Convergence of Environment-Dependent Models

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Let $\mathbb{Z}^d$ be a d-dimensional lattice, and each site on $\mathbb{Z}^d$ is occupied by either one of the two species. At each random time, a particle dies and is replaced by a new one, but the random time and the type chosen of the species are assumed to be determined by the environment conditions around the particle. The random function $\eta_t : \mathbb{Z}^d \to \{0,1\}$ denotes the state at time t, and each number of $\{0,1\}$ denotes the label of the type chosen of the two species. When we set $\|y\|_\infty := \max_i y_i$, we define $\mathcal{N}_x := x+ \{y : 0 < \|y\|_\infty \leqslant r\}$. For $i = 0,1$, let $f_i(x,\eta)$ be a frequency of type i in the neighborhood $\mathcal{N}_x$ of x for η. For non-negative parameters $\alpha_{ij} \geq 0$, the dynamics of η_t is defined as follows. The state η makes transition $0 \to 1$ at rate $\lambda f_1(f_0+\alpha_{01}f_1)/(\lambda f_1+f_0)$, and it makes transition $1 \to 0$ at rate $f_0(f_1+\alpha_{10}f_0)/(\lambda f_1+f_0)$. The exchange of particles after death is described in the form being proportional to the weighted density between the two species, expressed by a parameter λ. For brevity's sake we shall treat a simple case $\lambda = 1$ only in what follows. For $N = 1,2,\ldots,$ let $m_N \in \mathbb{N}$, and we put $\ell_N := m_N\sqrt{N}$, and $\mathbb{S}_N := \mathbb{Z}^d/\ell_N$. While, $W_N = (W_N^1, \ldots, W_N^d) \in (\mathbb{Z}^d/m_N) \setminus \{0\}$ is defined as a random vector satisfying (i) $\mathcal{L}(W_N) = \mathcal{L}(-W_N)$; (ii) $E(W_N^i W_N^j) \to \delta_{ij}\sigma^2(\geq 0)$ (as $N \to \infty$); (iii) $\{|W_N|^2\}$ $(N \in \mathbb{N})$ is uniformly integrable. Here $\mathcal{L}(Y)$ indicates the law of a random variable Y. For the kernel $p_N(x) := P(W_N/\sqrt{N} = x)$, $x \in \mathbb{S}_N$ and $\eta \in \{0,1\}^{\mathbb{S}_N}$, we define the scaled frequency f_i^N as

$$f_i^N(x,\eta) = \sum_{y\in\mathbb{S}_N} p_N(y-x)1_{\{\eta(y)=i\}}, \qquad (i = 0,1). \tag{1}$$

We denote by η_t^N the state determined by the scaled frequency depending on α_i^N and p_N. On this account, we may define the associated measure-valued process as

$$X_t^N := \frac{1}{N}\sum_{x\in\mathbb{S}_N} \eta_t^N(x)\delta_x. \tag{2}$$

For the initial value X_0^N, we assume that $\sup_N\langle X_0^N, 1\rangle < \infty$ and $X_0^N \to X_0$ in $M_F(\mathbb{R}^d)$ as $N \to \infty$, where $M_F(\mathbb{R}^d)$ is the totality of all the finite measures on $\mathbb{R}^d$, equipped with the topology of weak convergence. Let $\{\xi_t^x\}$ be a continuous time random walk with rate N and step distribution p_N starting at a point $x \in \mathbb{S}_N$, and $\{\hat{\xi}_t^x\}$ be a continuous time coalescing random walk with rate N and step distribution p_N starting at a point x. For a finite set $A \subset \mathbb{S}_N$, we denote by $\tau(A)$ the time when all the particles starting from A finally coalesce into a single particle, that is to say, we define $\tau(A) := \inf\{t > 0 : \#\{\hat{\xi}_t^x;\ x \in A\} = 1\}$. Take a sequence $\{\varepsilon_N\}$ of positive numbers such that $\varepsilon_N \to 0$ and $N\varepsilon_N \to \infty$ as $N \to \infty$. Moreover, we suppose that when $N \to \infty$,

$$N \cdot P(\xi^0_{\varepsilon_N} = 0) \to 0 \quad \text{and} \quad \sum_{e \in \mathbb{S}_N} p_N(e) \cdot P(\tau(\{0, e\}) \in (\varepsilon_N, t]) \to 0 \quad (\forall t > 0). \tag{3}$$

We also assume now that the following limits exist :

$$\lim_{N\to\infty} \sum_{e \in \mathbb{S}_N} p_N(e) \cdot P(\tau(\{0, e\}) > \varepsilon_N) = \exists\gamma(> 0) \tag{4}$$

$$\text{and} \qquad \lim_{N\to\infty} P\left(\tau(A/\ell_N) \leqslant \varepsilon_N\right) = \exists\zeta(A) \tag{5}$$

holds for any finite subset $A \subset \mathbb{Z}^d$.

Theorem. (cf.[1],[2]) When we denote the law of a measure-valued stochastic process $X^N_\cdot$ on the path space Ω_D by P_N, then there exists a probability measure $P^* \in \mathcal{P}(\Omega_C)$ such that

$$P_N \quad \Longrightarrow \quad P^*_{X_0} \qquad (\text{as} \quad N \to \infty). \tag{6}$$

Then there exists a $M_F(\mathbb{R}^d)$-valued stochastic process $X_t = X_t^{2\gamma,\theta,\sigma^2}$ named a DW superprocess with parameters 2γ, θ and σ^2, satisfying that X_t^N converges to $X_t^{2\gamma,\theta,\sigma^2}$ as $N \to \infty$ in the sense of weak convergence for measures, and $P^*_{X_0}$ is the law of $X_t^{2\gamma,\theta,\sigma^2}$.

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Phase transitions in a control problem

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We consider the following controlled system of SDE

$$\mathrm{d}X_t = \gamma \mathrm{d}W_t + \mathrm{d}\Lambda_t,$$
$$\mathrm{d}Y_t = X_t \mathrm{d}W_t + \mathrm{d}\|\Lambda\|_t - \beta \mathrm{d}t,$$

where $\beta > 0$ and $\gamma \neq 0$ are constants, W is a standard Brownian motion and $\|\Lambda\|$ is the total variation of our control Λ that we require to be an adapted process of finite variation. The problem, motivated by a financial practice of hedging under transaction costs, is to minimize

$$\limsup_{T\to\infty} \frac{1}{T} E[Y_T^2] = \limsup_{T\to\infty} \frac{1}{T} E[\int_0^T (X_t^2 - 2\beta Y_t)\mathrm{d}t + 2\int_0^T Y_t \mathrm{d}\|\Lambda\|_t].$$

This is a 2 dimensional stochastic control problem which is degenerate (both the Brownian motion W and the control Λ are only one dimensional). The optimal control is, by a formal dynamic programming principle, expected to be a singular one which keeps (X, Y) inside a region; however we have not yet had satisfactory results both from theoretical and practical points of view for this original problem. In this talk, we focus on a restricted class of control

$$\mathrm{d}\Lambda = -\mathrm{sgn}(X_t)\gamma^2 c(X_t)\mathrm{d}t + \mathrm{d}L_t - \mathrm{d}R_t,$$

where c is a nonnegative continuous even function on an interval $[-b, b]$, $b > 0$ and L and R are nondecreasing processes with

$$\mathrm{d}L_t = 1_{\{X_t=-b\}}\mathrm{d}L_t, \quad \mathrm{d}R_t = 1_{\{X_t=b\}}\mathrm{d}R_t \tag{1}$$

which keep X stay in $[-b, b]$. Now our control is (b, c). The idea of the control is to push X towards 0. The absolutely continuous part of Λ determined by c pushes X regularly towards 0. The other parts, that turn out to be singularly continuous, are active only when X reaches the boundary of $[-b, b]$ and push X to prevent it from going out of the interval. Such a control exists; in fact, there exists a pathwise unique strong solution (X, L, R) of a Skorokhod SDE

$$\mathrm{d}X_t = \gamma \mathrm{d}W_t - \mathrm{sgn}(X_t)\gamma^2 c(X_t)\mathrm{dt} + \mathrm{d}L_t - \mathrm{d}R_t$$

on $[-b, b]$ when $x \mapsto -\mathrm{sgn}(x)c(x)$ is one-sided Lipschitz. The control Λ is then well-defined by (1). The optimal control in this restricted class is probably suboptimal for the original problem; however it has a certain advantage in its easy implementation. Also this type of control strategies has appeared in a related context of optimal hedging. Now, we are no more sure about the validity of the dynamic programming principle due to the constraint that the control Λ can only refer to the spot value of X. Although we can formally derive an HJB type equation, it is far from a standard form and difficult to solve. Here we present our results based on a probabilistic approach.

Theorem:

1. As $T \to \infty$,
$$\frac{1}{\sqrt{T}}(Y_T - \delta^{b,c}T) \to \mathcal{N}(0, Q^{b,c})$$
in law, where
$$\delta^{b,c} = \frac{\gamma^2}{a} - \beta, \ \ a = 2\int_0^b g(x)\mathrm{d}x, \ \ g(x) = \exp\left\{-2\int_0^x c(y)\mathrm{d}y\right\}$$
and
$$Q^{b,c} = \frac{2}{a}\int_0^b (x - \gamma h(x))^2 g(x)\mathrm{d}x, \ \ h(x) = \frac{2}{g(x)}\int_0^x \left(c(y) - \frac{1}{a}\right)g(y)\mathrm{d}y.$$

2. $$\lim_{T\to\infty}\frac{1}{T}E[(Y_T - \delta^{b,c}T)^2] = Q^{b,c}.$$

3. $$\inf_{\delta^{b,c}=0} Q^{b,c} = \gamma^2\eta\left(\frac{\gamma}{\beta}\right),$$
where
$$\eta(x) = \begin{cases} 0 & \text{if } -2 < x \le 1, \\ \frac{4}{3}\frac{(x+2)^2(x-1)}{x^3(4-x)} & \text{if } 1 < x < 2, \\ \frac{1}{12}(x+2)^2 & \text{if } |x| \ge 2. \end{cases}$$

The proof of the convergences is not difficult. The mathematically challenging part is the minimization of $Q^{b,c}$. We can give an explicit sequence of controls (b_n, c_n) with $\delta^{b_n,c_n} = 0$ such that Q^{b_n,c_n} converges to the infimum. In fact, $b_n = \gamma^2/2\beta$ and $c_n = 0$ when $|\gamma| \ge 2\beta$. When $|\gamma| < 2\beta$ on the other hand, $b_n \to \infty$ as $n \to \infty$ and the pointwise limit of $c_n(x)$ is given by
$$c_\infty(x) = \frac{\gamma + 2\beta}{2(\gamma - \beta l)}\frac{1}{\gamma + |x|}1_{\{|x|\ge l\}}, \ \ l = \frac{2(\gamma-\beta)_+}{4\beta - \gamma}.$$
The key for the minimization is to show
$$\inf_{y\in\mathcal{Y}_a}\int_0^1 \left(y(u) + \gamma + \frac{2\gamma}{a}(u-1)y'(u)\right)^2 \mathrm{d}u = \gamma^2\eta(a/\gamma),$$
where $\mathcal{Y}_a$ is the set of the convex functions on $[0, 1]$ with $y(0) = 0$ and $y'(0) = a/2$.

The rates of the L^p-convergence of the Euler-Maruyama and the Wong-Zakai approximations of path-dependent stochastic differential equations under the Lipschitz condition

Seiichiro Kusuoka
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In this talk, we consider the Euler-Maruyama and the Wong-Zakai approximations of path-dependent stochastic differential equations. We remark that the theorems below are applicable to the Markov type stochastic differential equations with reflecting boundary condition on sufficiently good domains.

First we consider the Euler-Maruyama approximation of path-dependent stochastic differential equations. Let $T > 0$ and let ξ be an $\mathbb{R}^d$-valued random variable. Consider the following stochastic differential equation

$$\begin{cases} dX_t &= \sigma(t,X)dB_t + b(t,X)dt \\ X_0 &= \xi \end{cases} \tag{1}$$

where σ is an $\mathbb{R}^d \otimes \mathbb{R}^r$-valued function on $[0,T] \times C_b([0,T];\mathbb{R}^d)$, b is an $\mathbb{R}^d$-valued function on $[0,T] \times C_b([0,T];\mathbb{R}^d)$ and B is the r-dimensional Brownian motion. We assume the Lipschitz continuity of the coefficients in the following sense.

$$\begin{aligned} |\sigma(t,w) - \sigma(t,w')|_{\mathbb{R}^d\otimes\mathbb{R}^r} + |b(t,w) - b(t,w')|_{\mathbb{R}^d} \leq K_T \|w - w'\|_{C([0,t];\mathbb{R}^d)}, \\ t \in [0,T],\ w, w' \in C([0,T];\mathbb{R}^d) \end{aligned} \tag{2}$$

where K_T is a constant depending on T. Then, the solution X to (1) exists, and has the pathwise uniqueness. Let $\triangle := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ be a partition of the interval $[0,T]$. Define the approximations $\sigma_\triangle, b_\triangle$ of σ, b by

$$\sigma_\triangle(t,w) := \sigma(t_k,w),\ b_\triangle(t,w) := b(t_k,w), \quad t \in [t_k, t_{k+1})$$

for $k = 0, 1, \ldots, N-1$, and $w \in C([0,T];\mathbb{R}^d)$. We consider the following stochastic differential equation.

$$\begin{cases} dX_t^{\mathrm{EM}} &= \sigma_\triangle(t,X^{\mathrm{EM}})dB_t + b_\triangle(t,X^{\mathrm{EM}})dt \\ X_0^{\mathrm{EM}} &= \xi. \end{cases} \tag{3}$$

Then, (3) is the equation of the Euler-Maruyama approximation to (1). For a Hilbert space H and a positive number K, we define a class of H-valued functions $F_K(H)$ by the total set of $h : [0,T] \times C_b([0,T];\mathbb{R}^d) \to H$ such that

(F1) $|h(t,w)|_H \leq K$ for $t \in [0,T],\ w \in C([0,T];\mathbb{R}^d)$.

(F2) $|h(t,w) - h(s,w)|_H \leq K(\sqrt{t-s} + \|w(\cdot + s) - w(s)\|_{C([0,t-s];\mathbb{R}^d)})$
for $s, t \in [0,T]$ such that $s < t$, and $w \in C([0,T];\mathbb{R}^d)$.

(F3) $|h(t,w) - h(t,w')|_H \leq K\|w - w'\|_{C([0,t];\mathbb{R}^d)}$ for $t \in [0,T],\ w, w' \in C([0,T];\mathbb{R}^d)$.

Then, we have the following theorem.

Theorem 1. *Let $\sigma \in F_K(\mathbb{R}^d \otimes \mathbb{R}^r)$ and $b \in F_K(\mathbb{R}^d)$. Let X and X^{EM} be the solutions to (1) and to the equation of the Euler-Maruyama approximation (3), respectively. Then, for $p \in [1,\infty)$ there exists a constant C independent of $\triangle$ and N, such that*

$$E\left[\|X - X^{\mathrm{EM}}\|_{C([0,T];\mathbb{R}^d)}^p\right]^{1/p} \leq C|\triangle|^{1/2}.$$

Next we consider the Wong-Zakai approximation of path-dependent stochastic differential equations. Let $T > 0$. Let A be a mapping from $C([0,T];\mathbb{R}^d)$ to $C([0,T];\mathbb{R}^d)$ such that

(A1) $\|A(w) - A(w')\|_{C([0,t];\mathbb{R}^d)} \le K_A \|w - w'\|_{C([0,t];\mathbb{R}^d)}$ for $t \in [0,T],\ w, w' \in C([0,T];\mathbb{R}^d)$.

(A2) $|A(w)_t - A(w)_s|_{\mathbb{R}^d} \le K_A \left(\sqrt{t-s} + \|w(\cdot + s) - w(s)\|_{C([0,t-s];\mathbb{R}^d)} \right)$
for $s, t \in [0,T]$ such that $s < t$, and $w \in C([0,T];\mathbb{R}^d)$.

(A3) $\mathrm{Var}_{[0,t]}(A(w)) \le K_A (1 + \|w - w(0)\|_{C([0,t];\mathbb{R}^d)})$ for $t \in [0,T],\ w \in C([0,T];\mathbb{R}^d)$.

where $\mathrm{Var}_{[0,t]}(w)$ is the total variation of w on $[0,t]$, and let $f \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ which has the bounded derivatives. Define the mapping $\Gamma : C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$ by

$$(\Gamma w)_t := f(t, w_t) + A(w)_t, \quad t \in [0,T],\ w \in C([0,T];\mathbb{R}^d). \tag{4}$$

Then, we have the Lipschitz continuity of Γ in the sense of (2). From (A2) and (4), we have

$$|(\Gamma w)_t - (\Gamma w)_s|_{\mathbb{R}^d} \le (K_f + K_A) \left(\sqrt{t-s} + \|w(\cdot + s) - w(s)\|_{C([0,t-s];\mathbb{R}^d)} \right) \tag{5}$$

for $s, t \in [0,T]$ such that $s < t$, and $w \in C([0,T];\mathbb{R}^d)$, where K_f is a constant depending on the bounds of f and the derivatives of f. Let $\sigma \in C_b([0,T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^r)$ such that $\sigma(t,x,y)$ is differentiable with respect to x and y, and σ and the derivatives are Lipschitz continuous. Let $b \in C_b([0,T] \times C([0,T];\mathbb{R}^d); \mathbb{R}^d)$ such that there exists a positive constant K_b satisfying

$$|b(t,w) - b(t,w')|_{\mathbb{R}^d} \le K_b \|w - w'\|_{C([0,t];\mathbb{R}^d)}$$

for $t \in [0,T]$, and $w, w' \in C([0,T];\mathbb{R}^d)$. Let ξ be an $\mathbb{R}^d$-valued random variable. Consider the following stochastic differential equation of the Stratonovich type

$$\begin{cases} dX_t &= \sigma(t, X_t, (\Gamma X)_t) \circ dB_t + b(t,X)dt \\ X_0 &= \xi. \end{cases} \tag{6}$$

For a given partition $\triangle := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of the interval $[0,T]$, we define the piecewise linear approximation $B^\triangle$ of B by

$$B_t^\triangle := B_{t_k} + \frac{t - t_k}{t_{k+1} - t_k}(B_{t_k+1} - B_{t_k}), \quad t \in [t_k, t_{k+1}).$$

We define the equation of the Wong-Zakai approximation to (6) by

$$\begin{cases} dX_t^{\mathrm{WZ}} &= \sigma(t, X_t^{\mathrm{WZ}}, (\Gamma X^{\mathrm{WZ}})_t) dB_t^\triangle + b(t, X^{\mathrm{WZ}})dt \\ X_0^{\mathrm{WZ}} &= \xi. \end{cases} \tag{7}$$

Then, we have the following theorem.

Theorem 2. *Let σ and b as above. Let X and X^{WZ} be the solutions to (6) and to the equation of the Wong-Zakai approximation (7), respectively. Then, for $p \in [1,\infty)$ there exists a constant C independent of $\triangle$ and N, such that*

$$E\left[\|X - X^{\mathrm{WZ}}\|_{C([0,T];\mathbb{R}^d)}^p \right]^{1/p} \le C|\triangle|^{1/2}(1 + \log N)^{1/2}.$$

Lectures on approximate-flows and rough flows

It was realized in the late 70's that stochastic differential equations not only define individual trajectories, they also define flows of regular homeomorphisms, depending on the regularity of the vector fields involved in the dynamics. This opened the door to the study of stochastic flows of maps for themselves, and it did not took long time before Le Jan and Watanabe clarified definitely the situation by showing that, in a semimartingale setting, there is a one-to-one correspondence between flows of diffeomorphisms and time-varying stochastic velocity fields, under proper regularity conditions on the objects involved. We offered in the work [1] an embedding of the theory of semimartingale stochastic flows into the theory of rough flows similar to the embedding of the theory of stochastic differential equations into the theory of rough differential equations.

It is based on the "approximate flow-to-flow" machinery introduced in [2], which gives body to the following fact. To a 2-index family $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$ of maps which falls short from being a flow, in a quantitative way, one can associate a unique flow $(\varphi_{ts})_{0\leqslant s\leqslant t\leqslant T}$ close to $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$; moreover the flow φ depends continuously on the approximate flow μ. The point about such a machinery is that approximate flows appear naturally in a number of situations as simplified descriptions of complex evolutions, often under the form of Taylor-like expansions of complicated dynamics. The model situation is given by a controlled ordinary differential equation

$$\dot{x}_t = \sum_{i=1}^{\ell} V_i(x_t)\, \dot{h}_t^i, \tag{1}$$

in $\mathbb{R}^d$, driven by an $\mathbb{R}^\ell$-valued $\mathcal{C}^1$ control h. The Euler scheme

$$\mu_{ts}(x) = x + \big(h_t^i - h_s^i\big)V_i(x)$$

defines, under proper regularity conditions on the vector fields, an approximate flow whose associated flow is the flow generated by equation (1). One step farther, if we are given a weak geometric Hölder p-rough path $\mathbf{X}$, with $2 \leqslant p < 3$, and sufficiently regular vector fields $\mathrm{F} = \big(V_1, \dots, V_\ell\big)$ on $\mathbb{R}^d$, one can associate to the rough differential equation

$$dx_t = \mathrm{F}(x_t)\mathbf{X}(dt), \tag{2}$$

some maps μ_{ts} defined, for each $0 \leqslant s \leqslant t \leqslant T$, as the time 1 map of an ordinary differential equation involving $\mathbf{X}_{ts}$, the V_i and their brackets, that have the same Taylor expansion as the awaited Taylor expansion of a solution flow to equation (2). They happen to define an approximate flow whose associated flow is the solution flow to equation (2).

A similar approach can be used to deal with a general class of stochastic time-dependent velocity fields. We introduced for that purpose in [1] a notion of rough driver, that is an enriched version of a time-dependent vector field, that will be given by the additional datum of a time-dependent second order differential operator satisfying some algebraic and analytic conditions. A notion of solution to a differential equation driven by a rough driver will be given, in the line of what was done in [2]for rough differential equations, and the approximate flow-to-flow machinery will be seen to lead to a clean and simple well-posedness result for such equations. As awaited from the above discussion, the main point of this result is that the Itô map, that associates to a rough driver the solution flow to its associated equation, is continuous. This continuity result is the key to deep results in the theory of stochastic flows.

We proved indeed in [1] that reasonable semimartingale velocity fields can be lifted to rough drivers under some mild boundedness and regularity conditions, and that the solution flow associated to the 'semimartingale' rough driver coincides almost surely with the solution flow to the Kunita-type Stratonovich differential equation driven by the velocity field. As a consequence of the continuity of the Itô map, a Wong-Zakai theorem was proved for a general class of semimartingale velocity fields, together with sharp support and large deviation theorems for Brownian flows.

These two lectures will introduce the audience to the core of the machinery of approximate and rough flows. The "approximate-flow-to-flow" machinery will be introduced in lecture 1, and used to get back the basics of Lyons' theory of rough differential equations. Lecture 2 will set the scene of rough drivers and rough flows, with hints as to how on can embed the theory of stochastic flows of homeomorphisms into the theory of rough flows.

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Convergence of Brownian motions on RCD spaces

Kohei Suzuki (Kyoto Univeristy)

1 Introduction & Result

In this talk, we consider the following problem:

(Q) Does **the weak convergence** of Brownian motions follow **only** from **geometrical convergence** of the underlying spaces (or, vice versa)?

As a main result in this talk, we show that the weak convergence of the laws of Brownian motions is **equivalent** to the measured Gromov–Hausdorff (mGH) convergence of the underlying metric measure spaces under the following assumption:

Assumption 1.1 Let N, K and D be constants with $1 < N < \infty$, $K \in \mathbb{R}$ and $0 < D < \infty$. For $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, let $\mathcal{X}_n = (X_n, d_n, m_n)$ be a metric measure space satisfying the $\mathrm{RCD}^*(K, N)$ condition with $\mathrm{Diam}(X_n) \le D$ and $m_n(X_n) = 1$.

Under Assumption 1.1, it is known that there exists a conservative Hunt process on $\mathcal{X}_n$ associated with the Cheeger energy and unique in all starting points in $\mathcal{X}_n$. We denote it by $(\{\mathbb{P}_n^x\}_{x\in X_n}, \{B_t^n\}_{t\ge 0})$, called *the Brownian motion on* $\mathcal{X}_n$. We state our main theorem precisely:

Theorem 1.2 *Suppose that Assumption 1.1 holds. Then the following statements (i) and (ii) are equivalent:*

(i) (mGH-convergence of the underlying spaces)

$\mathcal{X}_n$ converges to $\mathcal{X}_\infty$ in the measured Gromov–Hausdorff sense.

(ii) (Weak convergence of the laws of Brownian motions)

There exist

$$\begin{cases} \text{a compact metric space } (X, d) \\ \text{isometric embeddings } \iota_n : X_n \to X \ (n \in \overline{\mathbb{N}}) \\ x_n \in X_n \ (n \in \overline{\mathbb{N}}) \end{cases}$$

such that

$$\iota_n(B_\cdot^n)_\# \mathbb{P}_n^{x_n} \to \iota_\infty(B_\cdot^\infty)_\# \mathbb{P}_\infty^{x_\infty} \quad \textit{weakly} \quad \textit{in } \mathcal{P}(C([0,\infty); X)).$$

The subscript $\#$ means the operation of the push-forward of measures.

$\mathbf{RCD}^*(\mathbf{K},\mathbf{N})$ (Riemannian Curvature-Dimension) spaces, introduced by Erbar–Kuwada–Sturm [2], are metric measure spaces satisfying a generalized notion of "$\mathbf{Ricci} \geq \mathbf{K}, \mathbf{dim} \leq \mathbf{N}$", which include several important classes of non-smooth spaces. For example, measured Gromov–Hausdorff (**mGH**) **limit spaces** of complete Riemannian manifolds with Ricci $\geq K, \dim = N$, or **Alexandrov spaces** with Curv $\geq K/(N-1), \dim = N$ are included in $\mathrm{RCD}^*(K,N)$ spaces.

Remark 1.3 We give comments to several related works.

(i) In [4], Ogura studied the weak convergence of the laws of the Brownian motions on Riemannian manifolds *by a different approach from this talk.* He push-forwarded all Brownian motions not to the ambient space X, but to *the limit space* M_∞ with respect to approximation maps $f_n : M_n \to M_\infty$ of the Kasue–Kumura convergence with certain *time-discretization* of Brownian motions.

More precisely, he assumed uniform upper bounds for heat kernels, and the Kasue–Kumura spectral convergence ([3]) of the underlying manifolds M_n. He push-forward each Brownian motions on M_n to the Kasue–Kumura spectral limit space M_∞ with respect to ε_n-isometry $f_n : M_n \to M_\infty$, and show the convergence in law on the càdlàg space of the push-forwarded and time-discretized Brownian motions on M_∞.

(ii) In [1], Albeverio and Kusuoka studied diffusion processes associated with SDEs on thin tubes in $\mathbb{R}^d$ shrinking to one-dimensional spider graphs. They studied the weak convergence of these diffusions to one-dimensional diffusions on the limit graphs. Their setting does not satisfy the $\mathrm{RCD}^*(K,N)$ condition because Ricci curvatures are not bounded below at points of conjunctions in spider graphs.

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KPZ EQUATION WITH FRACTIONAL DERIVATIVES OF WHITE NOISE

MASATO HOSHINO

We discuss the stochastic partial differential equation

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + (\partial_x h(t,x))^2 + \partial_x^\gamma \xi(t,x) \tag{1}$$

for $(t,x) \in [0,\infty) \times \mathbb{T}$ with $\gamma \geq 0$. Here, ξ is a space-time white noise on $[0,\infty) \times \mathbb{T}$ and $\partial_x^\gamma = -(-\partial_x^2)^{\frac{\gamma}{2}}$ is the fractional derivative. When $\gamma = 0$, this equation is called KPZ equation, which is proposed in [3] as a model of surface growth. Hairer discussed the solvability of KPZ equation in [1]. He showed in [1] that the renormalized equation

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \xi_\epsilon(t,x), \tag{2}$$

where ξ_ϵ is a smooth approximation of ξ and $C_\epsilon \sim \frac{1}{\epsilon}$ is a sequence of constants, has a unique limiting process h, which is independent of the way to approximate ξ.

Our goal is to make the noise rougher and see to what extent this theory works. Because of the "local subcriticality" ([2]), we can expect that the similar results hold if $\gamma < \frac{1}{2}$. However, we show that the renormalization like (2) is possible only for $0 \leq \gamma < \frac{1}{4}$.

Theorem 1. *Let $\rho = \rho(t,x)$ be a function on $\mathbb{R}^2$ which is smooth, compactly supported, symmetric in x, nonnegative, and satisfies $\int_{\mathbb{R}^2} \rho(t,x)dtdx = 1$. Let $0 \leq \gamma < \frac{1}{4}$ and $0 < \alpha < \frac{1}{2} - \gamma$. Then there exists a sequence of constants C_ϵ such that*

1. *We have $C_\epsilon \leq C\epsilon^{-1-2\gamma}$ for some constant C (depending on γ and ρ).*
2. *For every initial condition $h_0 \in \mathcal{C}^\alpha(\mathbb{T})$, the sequence of solutions h_ϵ to the equation:*
$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \partial_x^\gamma \xi_\epsilon(t,x)$$
on $(t,x) \in [0,T) \times \mathbb{T}$ for some random time T, converges to a unique stochastic process h, which is independent of the choice of ρ.

This convergence holds in probability in the uniform norm on all compact sets in $[0,T) \times \mathbb{T}$ and α-Hölder norm on all compact sets in $(0,T) \times \mathbb{T}$.

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Existence and uniqueness of strict solutions of stochastic linear evolution equations in M-type 2 Banach spaces

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(joint work with Yoshitaka Yamamoto and Atsushi Yagi)

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1. Introduction

We study non-autonomous abstract stochastic evolution equations of the form

$$\begin{cases} dX + A(t)X\,dt = F(t)dt + G(t)dW(t), & 0 < t \le T, \\ X(0) = \xi, \end{cases} \tag{1.1}$$

in a complex separable Banach space $(E, \|\cdot\|)$ of M-type 2. Here, $\{A(t), t \ge 0\}$ is a family of densely defined, closed linear operators in E; $W(t)$ is a cylindrical Wiener process on separable Hilbert space U and is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$; F is an E-valued predictable function; G is an $L_2(U;E)$-valued predictable function where $L_2(U;E)$ denotes the space of Hilbert-Schmidt operators; and ξ is an $\mathcal{F}_0$-measurable random variable. We suppose that A, F and G satisfy the following structural assumptions.

(A1) For all $t \in [0,T]$, the spectrum $\sigma(A(t))$ and the resolvent of $A(t)$ satisfy

$$\sigma(A(t)) \subset \Sigma_\varpi = \{\lambda \in \mathbb{C} : |\arg \lambda| < \varpi\}$$

and

$$\|(\lambda - A(t))^{-1}\| \le \frac{M_\varpi}{|\lambda|}, \qquad \lambda \notin \Sigma_\varpi$$

with some constants $\varpi \in (0, \frac{\pi}{2})$ and $M_\varpi > 0$.

(A2) There exists an exponent $\nu \in (0,1]$ such that

$$\mathcal{D}(A(s)) \subset \mathcal{D}(A(t)^\nu), \qquad 0 \le s,t \le T.$$

(A3) There exist an exponent $\mu \in (1-\nu, 1]$ and a constant $N > 0$ such that

$$\|A(t)^\nu [A(t)^{-1} - A(s)^{-1}]\| \le N|t-s|^\mu, \qquad 0 \le s,t \le T.$$

(F1) There exist $\beta \in (0,1]$ and $0 < \sigma < \min\{\beta, \mu+\nu-1\}$ such that $F \in \mathcal{F}^{\beta,\sigma}((0,T];E)$ a.s., where $\mathcal{F}^{\beta,\sigma}((0,T];E)$ denotes the weighted Hölder continuous function space.

(G1) There exist a constant $\delta > \frac{1}{2}$ and a square-integrable random variable ζ such that

$$\|A(t)^\delta G(t) - A(s)^\delta G(s)\|_{L_2(U;E)} \le \zeta |t-s|^\sigma \qquad \text{a.s.}$$

and $\mathbb{E}\|A(0)^\delta G(0)\|^2_{L_2(U;E)} < \infty$.

2. Main results

Theorem 2.1 (Uniqueness [1]). *Let* (A1), (A2), (A3) *be satisfied. If there exists a strict solution to the equation* (1.1) *then it is unique.*

Theorem 2.2 (Existence [1]). *Let* (A1), (A2), (A3), (F1) *and* (G1) *be satisfied. Suppose that* $\xi \in \mathcal{D}(A(0)^\beta)$ *a.s. and* $\mathbb{E}\|A(0)^\beta \xi\|^2 < \infty$. *Then there exists a unique strict solution of* (1.1) *possessing the regularity:*

$$A^\beta X \in \mathcal{C}([0,T];E), \quad X \in \mathcal{C}^{\gamma_1}([0,T];E) \qquad a.s.$$

and

$$AX \in \mathcal{C}^{\gamma_2}([\epsilon,T];E) \qquad a.s.$$

for every $0 < \gamma_1 < \min\{\beta, \frac{1}{2}\}$, $0 < \gamma_2 < \min\{\delta - \frac{1}{2}, \sigma\}$ *and* $\epsilon \in (0,T]$. *In addition,* X *satisfies the estimate*

$$\begin{aligned}\mathbb{E}\|A^\beta X(t)\|^2 \leq & C[\mathbb{E}\|A(0)^\beta \xi\|^2 + \mathbb{E}\|F\|^2_{\mathcal{F}^{\beta,\sigma}} \\ & + \mathbb{E}\|A(0)^\delta G(0)\|^2_{L_2(U;E)} t^{1-2(\beta-\delta)} + t^{1-2(\beta-\delta)+2\sigma}]\end{aligned}$$

for the case $\beta \geq \delta$ *and*

$$\mathbb{E}\|A^\beta X(t)\|^2 \leq C[\mathbb{E}\|A(0)^\beta \xi\|^2 + \mathbb{E}\|F\|^2_{\mathcal{F}^{\beta,\sigma}} + \mathbb{E}\|A(0)^\delta G(0)\|^2_{L_2(U;E)} t + t^{1+2\sigma}]$$

for the case $\beta < \delta$. *Furthermore, if* $A(0)^\delta G(0) = 0$ *then*

$$X \in \mathcal{C}^{\gamma_1}([0,T];E) \qquad \textit{for every } 0 < \gamma_1 < \min\{\frac{1+\sigma}{2}, \delta, \beta\}.$$

In the favorable case $\nu = 1$ (see (A2)), the condition (G1) in Theorem 2.2 can be replaced by a simplified one, say

(G1)' There exist a constant $\delta_1 > \dfrac{1}{2}$ and a square-integrable random variable $\bar{\zeta}$ such that

$$\|A(0)^{\delta_1}[G(t) - G(s)]\|_{L_2(U;E)} \leq \bar{\zeta}|t-s|^\sigma \qquad \text{a.s.}$$

and $\mathbb{E}\|A(0)^{\delta_1} G(t)\|^2_{L_2(U;E)} < \infty$ for every $t \in [0,T]$.

Theorem 2.3 ([1]). *If* (G1)' *takes place then so does* (G1).

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On density function concerning discrete time maximum of some one-dimensional diffusion processes

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1 Introduction

In this talk, we will show some results on the density functions related to discrete time maximum of some one-dimensional diffusion processes. That is defined by $M_T^n = \max\{X_{t_1}, \cdots, X_{t_n}\}$ for a fixed time interval $[0,T]$ and a time partition $\Delta_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < t_{n+1} = T$ for $n \geq 2$, where $\{X_t, t \in [0,\infty)\}$ denotes a one-dimensional diffusion process.

Firstly, we shall deal with the following one-dimensional stochastic differential equation (SDE),

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, t \in [0,\infty) \tag{1}$$

where $x_0 \in \mathbb{R}$, $b, \sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions and $\{W_t, t \in [0,\infty)\}$ is a one-dimensional standard Brownian motion.

The first goal of this talk is to prove an integration by parts (IBP) formula for the random vector (M_T^n, X_T). That is the formula of the form $E[\partial_\beta \varphi(M_T^n, X_T)] = E[\varphi(M_T^n, X_T)H_\beta]$ for a smooth function φ, where H_β is a certain random variable and $E[\cdot]$ denotes the expectation with respect to a certain probability measure. Then, we will apply the IBP formula to study on the density function of (M_T^n, X_T).

The second goal is to obtain asymptotic behaviors of the density functions of M_T^n and (M_T^n, X_T) for Gaussian processes. For this purpose, we shall consider the following multiple integral,

$$I(\theta) := \int_R f(x_1, \cdots, x_n) e^{-\theta^2 \phi(x_1, \cdots, x_n) + k(\theta)\psi(x_1, \cdots, x_n)} dx_1 \cdots dx_n, \tag{2}$$

where $R = \prod_{i=1}^n (-\infty, d_i]$, $d_i \in \mathbb{R}$ for $1 \leq i \leq n$ and $f, \phi, \psi : \mathbb{R}^n \to \mathbb{R}$ are measurable functions, then obtain the asymptotic behavior of $I(\theta)$ as $\theta \to \infty$ by using the Laplace's method. The result will be used to obtain the asymptotic behaviors of the density functions. The process satisfying (1) where b, σ do not depend on the space parameter, Brownian Bridge and Ornstein-Uhlenbeck process will be considered as the examples.

2 Main results

For b, σ of (1), we assume the following,

Assumption (A)

(A1) For $t \in [0,\infty)$, $b(t,\cdot), \sigma(t,\cdot) \in C_b^\infty(\mathbb{R};\mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t,\cdot)$ and $\sigma(t,\cdot)$ do not depend on t. In particular, let $c(\sigma)$ be a constant which bounds $|\sigma(t,x)|$.

(A2) There exists $c > 0$ such that

$$|\sigma(t,x)| \geq c$$

holds, for any $x \in \mathbb{R}$ and $t \in [0,\infty)$.

Theorem 1. *Assume (A). Let $G \in \mathbb{D}^\infty$. Then, for any multi index $\beta \in \{1,2\}^k$, $k \geq 1$, there exists $H_\beta(G) \in \mathbb{D}^\infty$ such that*

$$E^P[\partial_\beta \varphi(M_T^n, X_T)G] = E^P[\varphi(M_T^n, X_T)H_\beta(G)] \tag{3}$$

holds for arbitrary $\varphi \in C_b^\infty(\mathbb{R}^2;\mathbb{R})$.

For $f, \phi, \psi, k(\theta)$ of (2), we assume the following,

Assumption (B)

(B1) $\phi \in C^2(\mathbb{R}^n;\mathbb{R})$ and ϕ attains its global minimum at a point $x^* = (x_1^*, \cdots, x_n^*) \in R$, in particular, we assume that $x_{j_1}^* = d_{j_1}, \cdots, x_{j_m}^* = d_{j_m}$ for $1 \leq j_1 < \cdots < j_m \leq n$, $0 \leq m \leq n$ and $x_i^* < d_i$ for other $1 \leq i \leq n$.

(B2) There exist $a_i > 0$ and $b_i \in \mathbb{R}$, $1 \leq i \leq n$ such that $\phi(x_1, \cdots, x_n) \geq \sum_{i=1}^n a_i x_i^2 + \sum_{i=1}^n b_i x_i$ holds.

(B3) $\psi \in C^1(\mathbb{R}^n;\mathbb{R})$ and there exist $c_i \geq 0$, $1 \leq i \leq n$ such that $\psi(x_1, \cdots, x_n) \leq \sum_{i=1}^n c_i |x_i|$ holds.

(B4) $f \in C^1(\mathbb{R}^n;\mathbb{R})$ and there exist $K_1 > 0$ and $\alpha_i \geq 0$, $1 \leq i \leq n$ such that $|f(x_1, \cdots, x_n)| \leq K_1 e^{\sum_{i=1}^n \alpha_i x_i^2}$ holds. Moreover, we assume that $f(x^*) \neq 0$.

(B5) $k(\theta) \geq 0$ and $k(\theta) = o((\log(\theta))^2)$ hold.

Since $Hess\phi(x^*)$ is a positive definite matrix, we may use the orthogonal matrix Q and the diagonal matrix Λ satisfying $Hess\phi(x^*) = Q\Lambda Q^T$ and we denote these components

$$Q = \begin{bmatrix} q_{1,1} & \cdots & q_{1,n} \\ \vdots & \ddots & \vdots \\ q_{n,1} & \cdots & q_{n,n} \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \tag{4}$$

where $\lambda_i > 0$, $1 \leq i \leq n$ denote the eigenvalues of $Hess\phi(x^*)$.

The main theorem in this section is following.

Theorem 2. *Assume (B). Define* $w = \int_{\mathcal{C}} e^{-\frac{1}{2}\sum_{i=1}^n x_i^2} dx$, *where* $\mathcal{C}$ *is given by*

$$\mathcal{C} = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \middle| \sum_{k=1}^n \frac{q_{j_i,k}}{\sqrt{\lambda_k}} x_k \leq 0 \; (1 \leq i \leq m) \right\}, \tag{5}$$

for $1 \leq m \leq n$ *and* $\mathcal{C} = \mathbb{R}^n$ *for* $m = 0$. *Then, we have*

$$I(\theta) \sim w \frac{f(x^*)}{|Hess\phi(x^*)|^{\frac{1}{2}}} \frac{e^{-\theta^2 \phi(x^*) + k(\theta)\psi(x^*) + \frac{k(\theta)^2}{2\theta^2} \sum_{i=1}^n \frac{1}{\lambda_i} (\sum_{j=1}^n \partial_i \psi(x^*) q_{j,i})^2}}{\theta^n}, \theta \to \infty. \tag{6}$$

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Error analysis for approximations to one-dimensional SDEs via perturbation method *

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1 Introduction and main result

For a one-dimensional fractional Brownian motion (fBm) B with the Hurst $1/3 < H < 1$, we consider a one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,d^\circ B_s, \quad t \in [0,1], \tag{1}$$

where $x_0 \in \mathbf{R}$ is a deterministic initial value and $d^\circ B$ stands for the symmetric integral in the sense of Russo-Vallois. In order to approximate the solution to (1), we consider the Crank-Nicholson scheme as real-valued stochastic process on the interval $[0,1]$. In this talk, we study asymptotic error distributions of the scheme.

In what follows, we assume that the coefficients b and σ in (1) are smooth and they are bounded together with all their derivatives. We give the definition of the Crank-Nicholson scheme for the m-th dyadic partition $\{\tau_k^m = k2^{-m}\}_{k=0}^{2^m}$:

Definition 1.1 (The Crank-Nicholson scheme)**.** For every $m \in \mathbf{N}$, the Crank-Nicholson scheme $X^{\mathrm{CN}(m)} : [0,1] \to \mathbf{R}$ is defined by a solution to an equation

$$\begin{cases} X_0^{\mathrm{CN}(m)} = x_0, \\ X_t^{\mathrm{CN}(m)} = X_{\tau_{k-1}^m}^{\mathrm{CN}(m)} + \dfrac{1}{2}\left\{ b(X_{\tau_{k-1}^m}^{\mathrm{CN}(m)}) + b(X_t^{\mathrm{CN}(m)}) \right\}(t - \tau_{k-1}^m) \\ \qquad\qquad + \dfrac{1}{2}\left\{ \sigma(X_{\tau_{k-1}^m}^{\mathrm{CN}(m)}) + \sigma(X_t^{\mathrm{CN}(m)}) \right\}(B_t - B_{\tau_{k-1}^m}) \quad \text{for } \tau_{k-1}^m < t \le \tau_k^m. \end{cases}$$

Since the Crank-Nicholson scheme is an implicit scheme, we need to restrict the domain of it and assure the existence of a solution to the equation above. Roughly speaking, the existence of the solution is ensured for large m.

In order to state our main result concisely, we set $\mathsf{w} = \sigma b' - \sigma' b$ and

$$J_t = \exp\left(\int_0^t b'(X_u)\,du + \int_0^t \sigma'(X_u)\,d^\circ B_u \right).$$

We assume the following hypothesis in order to obtain an expression of the error of the scheme:

*This talk is based on a joint work with Professor Shigeki Aida.

Hypothesis 1.2. $\inf \sigma > 0$.

The following is our main result:

Theorem 1.3. *Assume that Hypothesis* 1.2 *is satisfied. For* $1/3 < H < 1/2$*, we have*

$$\lim_{m\to\infty} 2^{m(3H-1/2)}\{X^{\mathrm{CN}(m)} - X\} = \sigma(X)U + J\int_0^{\cdot} J_s^{-1}\mathsf{w}(X_s)U_s\,ds$$

weakly with respect to the uniform norm. Here U *a stochastic process defined by*

$$U_t = \sigma_{3,H}\int_0^t f_{0,3}(X_u)\,dW_u, \tag{2}$$

where $\sigma_{3,H}$ *is a positive constant,* $f_{0,3} = (\sigma^2)''/24$ *and* W *is a standard Brownian motion independent of* B.

2 Sketch of proof

We explain the concept of perturbation method and give a sketch of proof of our main theorem.

The idea of perturbation method is to find a piecewise linear stochastic process $\tilde{h} \equiv \tilde{h}^{(m)} : [0,1] \to \mathbf{R}$ such that $X^{x_0,B+\tilde{h}}_{\tau_k^m} = X^{\mathrm{CN}(m)}_{\tau_k^m}$ for every $k = 1,\dots,2^m$, where $X^{x_0,B+\tilde{h}}$ is a solution to an SDE with the same initial value x_0 and a perturbed driver $B+\tilde{h}$, that is,

$$X^{x_0,B+\tilde{h}}_t = x_0 + \int_0^t b(X^{x_0,B+\tilde{h}}_s)\,ds + \int_0^t \sigma(X^{x_0,B+\tilde{h}}_s)\,d^\circ(B+\tilde{h})_s.$$

Under Hypothesis 1.2, we see unique existence of $\tilde{h}$ and obtain an expression of it.

From the expression of $\tilde{h}^{(m)}$ and the Lipschitz continuity of the solution map $B \mapsto X^{x_0,B}$, we construct a piecewise linear function $h \equiv h^{(m)} : [0,1] \to \mathbf{R}$ such that (a) $2^{m(3H-1/2)}h^{(m)}$ converges to U defined by (2) and (b) $\tilde{h}^{(m)} - h^{(m)}$ is negligible. We can show Assertion (a) by using the fourth moment theorem. Assertion (b) is a nontrivial part in our proof. In order to justify Assertion (b), we need the following step:

(D1) estimate $\delta^{(m)} = \max_{1\le k\le 2^m}|X^{\mathrm{CN}(m)}_{\tau_k^m} - X^{x_0,B}_{\tau_k^m}|$ from the definition of the scheme,

(H1) estimate $\|\tilde{h}^{(m)} - h^{(m)}\|_\infty$ by a quantity involving $\delta^{(m)}$ from the construction of $\tilde{h}^{(m)}$ and $h^{(m)}$,

(D2) estimate $\delta^{(m)}$ by a quantity involving $\delta^{(m)}$ itself from (H1),

(D3) show a sharp estimate of $\delta^{(m)}$ by using (D2) repeatedly and (D1),

(H2) show Assertion (b) from (D3) and (H1).

For simplicity, we explain how to see the asymptotic error distribution of $X_1^{\mathrm{CN}(m)} - X_1^{x_0,B}$. By using the properties of $h^{(m)}$ and the decomposition

$$\begin{aligned} X_1^{\mathrm{CN}(m)} - X_1^{x_0,B} &= X_1^{x_0,B+\tilde{h}^{(m)}} - X_1^{x_0,B} \\ &= \nabla_{h^{(m)}}X_1^{x_0,B} + \{X_1^{x_0,B+\tilde{h}^{(m)}} - X_1^{x_0,B+h^{(m)}}\} + \{X_1^{x_0,B+h^{(m)}} - X_1^{x_0,B} - \nabla_{h^{(m)}}X_1^{x_0,B}\}, \end{aligned}$$

we see Theorem 1.3. In fact, Assertion (a) implies that the first term converges to a nontrivial process, that is, $2^{m(3H-1/2)}\nabla_{h^{(m)}}X_1^{x_0,B} = \nabla_{2^{m(3H-1/2)}h^{(m)}}X_1^{x_0,B} \to \nabla_U X_1^{x_0,B}$ as $m \to \infty$. The convergences of the second and third term to 0 follow from Assertion (a) and (b), respectively.

Parametrix method for skew diffusion

Dai Taguchi (Ritsumeikan University)
joint work with
Arturo Kohatsu-Higa (Ritsumeikan University)
Jie Zhong (University of Central Florida)

A skew diffusion is the unique solution of the following one-dimensional stochastic differential equation with local time:

$$X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \sigma(X_s(x))dW_s + (2\alpha - 1)L_t^0(X), t \in [0,T], \alpha \in (0,1), \quad (1)$$

where W is a one-dimensional standard Brownian motion. The stochastic process $L^0(X)$ is a symmetric local time of X at the origin, that is $L_t^0(X)$ is defined by

$$L_t^0(X) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[-\varepsilon,\varepsilon]}(X_s) d\langle X \rangle_s.$$

The simplest example of skew diffusion process is a skew Brownian motion which is the solution of (1) with $b = 0$ and $\sigma = 1$. Harrison and Shepp [3] proved that if $|2\alpha - 1| \leq 1$ then there is a unique strong solution and if $|2\alpha - 1| > 1$ and $x_0 = 0$, there is no solution. The idea of the proof is a transformation technique to relate (1) with another stochastic differential equation without local time and with discontinuous diffusion coefficient.

In this talk, we prove that the existence and a Gaussian upper bound for the density of a skew diffusion. The idea of proof is the parametrix method for the semigroup $P_t f(x) := \mathbb{E}[f(X_t(x))]$ which is a "Taylor-like expansion". Using the "Backward" parametrix method which is introduced in [2] and [1], we prove the expansion for the semigroup of SDE associated to (1) and its density, under the condition that the drift coefficient is bounded, measurable and the diffusion coefficient is bounded, uniformly elliptic and Hölder continuous. We also obtain the similar expansion for skew diffusion.

In this talk, we also consider a probabilistic representation which can be used Monte Carlo simulation and/or infinite dimensional analysis. More precisely, the parametrix expansion for the density of X_T, $p_T(x,.)$, leads to that for given $p \geq 1$, there exists a random variable $H(T,x,y)$ such that for any $(x,y) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$,

$$p_T(x,y) = \mathbb{E}[H(T,x,y)] \text{ and } \mathbb{E}[|H(T,x,y)|^p] < \infty.$$

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Regularization of Generalized Wiener Functionals by Bochner Integral

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The local time (at zero) of the one-dimensional Wiener process $w = (w(t))_{t\geq 0}$ starting from zero is heuristically written as

$$\text{“}\int_0^t \delta_0(w(s))\mathrm{d}s\text{”} \tag{1}$$

and is rigorously formulated by

$$\lim_{\varepsilon\to 0}\int_0^t \varphi_\varepsilon(w(s))\mathrm{d}s$$

with using rapidly decreasing functions $\{\varphi_\varepsilon\}_{\varepsilon>0}$ which tends to Dirac's delta function δ_0 in the space of Schwartz distributions.

In this talk, we formulate (1) directly as a Bochner integral. It is well known that $\delta_0(w(t))$ makes sense as a generalized Wiener functional (see Watanabe [4]) and belongs to $\mathbb{D}_2^{(-1/2)-\varepsilon}$ for each $\varepsilon > 0$. (see Nualart-Vives [2], Watanabe [5, 6]). For our objective, we need to consider the Bochner integrability of the mapping

$$(0,t] \ni s \mapsto \delta_0(w(s)) \in \mathbb{D}_2^{(-1/2)-\varepsilon}. \tag{2}$$

Note that $\delta_0(w(0))$ does not make sense as a generalized Wiener functional, and hence the Bochner integrability does not follow only from the continuity of the mapping $t \mapsto \delta_0(w(t))$.

When succeeded in seeing the Bochner integrability of the mapping (2), the Bochner integral $\int_0^t \delta_0(w(s))\mathrm{d}s$ makes sense as an element in $\mathbb{D}_2^{-(1/2)-\varepsilon}$ for $\varepsilon > 0$. However, the local time is known to be a classical Wiener functional, so that it should be in $\mathbb{D}_2^0 = L_2$. Hence, the Bochner integral should pose a sort of “regularizing effect”. This phenomenon might be a common understanding at the level of intuition for most of us, but there have not been literatures on this subject except for the case of local times.

Denote by $\mathscr{S}'(\mathbb{R})$ the space of Schwartz distributions on $\mathbb{R}$. The following is the prototype of this study:

Theorem 1. *Let $\Lambda \in \mathscr{S}'(\mathbb{R})$, $t > 0$ and $s \in \mathbb{R}$. If $\Lambda(w(t)) \in \mathbb{D}_2^s$ then the mapping*

$$(0,t] \ni u \mapsto \sqrt{\frac{t}{u}}\Lambda\Big(\sqrt{\frac{t}{u}}w(u)\Big) \in \mathbb{D}_2^s$$

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is Bochner integrable in $\mathbb{D}_2^s$ *and we have*

$$\int_0^t \sqrt{\frac{t}{u}}\Lambda\Big(\sqrt{\frac{t}{u}}w(u)\Big)\mathrm{d}u \in \mathbb{D}_2^{s+1}.$$

From this we obtain $\int_0^t \delta_0(w(u))\mathrm{d}u \in \mathbb{D}_2^{(1/2)-\varepsilon}$ for each $\varepsilon > 0$ which agrees with the results in [2, 3, 6].

The proof of the above theorem is obtained by looking at all chaos appearing in the Itô-Wiener expansion for $\Lambda((t/u)^{1/2}w(u))$ and relies on their explicit forms, so that it is hard to obtain a similar result for diffusion processes.

We will consider a similar "regularizing effect" for $\int_0^t \delta_0(X(s,x,w))\mathrm{d}s$ from purely Malliavin calculus-viewpoint, where $X(t,x,w)$ is the unique strong solution to a one-dimensional stochastic differential equation

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}w(t) + b(X_t)\mathrm{d}t.$$

Although we have not succeed to obtain a similar result to above, we can show the Bochner integrability of the mapping $u \mapsto \delta_0(X(u,x,w))$ and that $\int_0^t \delta_0(X(u,x,w))\mathrm{d}u \in L_2$. The proof is based on Itô's formula for generalized Wiener functionals which is a slight extension of Kubo [1].

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Large deviations for rough path lifts of Donsker–Watanabe's delta functions

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In 1993 Takanobu and Watanabe presented a large deviation principle (LDP) of Freidlin–Wentzell type for solutions of stochastic differential equations (SDEs) under the strong Hörmander condition anywhere. Unlike in the usual LDP of this type, the probability measures are not the push-forwards of the (scaled) Wiener measure, but the push-forwards of the measures of finite energy which is defined by the composition of the solutions of SDEs and the delta functions (i.e., Watanabe's pullbacks of the delta functions, also known as Donsker's delta function). One interpretation of this LDP is a generalization of the LDP of Freidlin–Wentzell type for pinned diffusion measures. This LDP looks very nice. To the author's knowledge, however, no proof has been given yet. In this talk we reformulate this LDP on the geometric rough path space by lifting these measures to the rough path sense and prove it rigorously by using quasi-sure analysis (which is a kind of potential theory in Malliavin calculus). Then, we obtain the LDP in Takanobu–Watanabe (1993) as a simple corollary of our main result. As a special case of this corollary, we also obtain the LDP for pinned diffusion measures under the strong Hörmander condition anywhere. (Even this one might be new.)