Large time asymptotics for Feynman-Kac functionals of symmetric stable processes

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Let $\{X_t\}$ be the rotationally invariant α-stable process on $\mathbb{R}^d$ with $0 < \alpha < 2$ and denote by $(\mathscr{E}, \mathscr{F})$ the corresponding Dirichlet form on $L^2(\mathbb{R}^d)$. We assume $\alpha < d$, transience of $\{X_t\}$ and denote the Green kernel by $G(x,y)$. Let μ be a positive Radon smooth measure satisfying Green-tightness and define the Schrödinger form $\mathscr{E}^\mu$ by $\mathscr{E}^\mu(u,v) = \mathscr{E}(u,v) - \langle u,v\rangle_\mu \equiv \langle -\mathscr{L}^\mu u, v\rangle$. Denoting by A_t^μ the positive continuous additive functional in the Revuz correspondence with μ, we have

$$\int_{\mathbb{R}^d} p^\mu(t,x,y)dy = \mathbb{E}_x[\exp(A_t^\mu)]. \tag{1}$$

Here $p^\mu(t,x,y)$ is the fundamental solution of the equation $\partial u/\partial t = \mathscr{L}^\mu u$. We call the right hand side of (1) *Feynman-Kac functional*. In this talk, we consider the large time asymptotics for $\mathbb{E}_x[\exp(A_t^\mu)]$. This is a jointly work with Professor Masayoshi Takeda.

We define the spectral bottom of the time changed process for $\{X_t\}$ by μ as follows:

$$\lambda(\mu) = \inf\{\mathscr{E}(u,u) \mid u \in \mathscr{F}_e, \quad \langle u,u\rangle_\mu = 1\},$$

where $\mathscr{F}_e$ is the extended Dirichlet space. Note that $\lambda(\mu)$ represents the smallness of μ. If $\lambda(\mu) > 1$, μ is said to be *subcritical*. Takeda [3] showed that μ is subcritical if and only if $\sup_{x\in\mathbb{R}^d} \mathbb{E}_x[\exp(A_\infty^\mu)] < \infty$. Moreover, if μ is of 0-order finite energy integral, this condition is also equivalent to the stability of fundamental solution, i.e. $p^\mu(t,x,y)$ admits the same two-sided estimates as the transition density function of $\{X_t\}$ up to positive multiple constants ([5]).

If $\lambda(\mu) < 1$, μ is said to be *supercritical*. The supercriticality of μ is equivalent to

$$C(\mu) := -\inf\{\mathscr{E}^\mu(u,u) \mid u \in \mathscr{F}, \quad \langle u,u\rangle = 1\} > 0$$

and this is the principal eigenvalue of $\mathscr{L}^\mu$. Via Fukushima's ergodic theorem, Takeda [4] showed $\mathbb{E}_x[\exp(A_t^\mu)] \sim c_1 h(x)\exp(C(\mu)t)$ where $h(x)$ is the eigenfunction corresponding to the principal eigenvalue.

If $\lambda(\mu) = 1$, μ is said to be *critical*. In this case $C(\mu) = 0$ and the growth of $\mathbb{E}_x[\exp(A_t^\mu)]$ is not exponential. Simon [2] and Cranston, Koralov et al. [1] treated the same problem for Brownian motion. They gave a concrete growth order of $\mathbb{E}_x[\exp(A_t^\mu)]$ depending on d for absolutely continuous μ with some additional conditions. For the proof, they first gave the asymptotic expansion of the β-order resolvent $G_\beta(x,y)$ as $\beta \to 0$ using the Hankel functions. The Schrödinger resolvent $\{G^\mu_\beta\}$ is expressed through G_β and the resolvent equation. Since it follows that $\mathbb{E}_x[\exp(A^\mu_t)] = 1 + \int_0^t P^\mu_s \mu ds$ for the Schrödinger semigroup $\{P^\mu_s\}$, their results follow via Tauberian theorem and the behavior of $G^\mu_\beta \mu$ as $\beta \to 0$.

In our framework, we impose only compactness on μ and thus need some improvements of their methods. First, we cannot express the resolvent kernel of the α-stable processes through special functions. The expression of the transition density function and some calculations enable us to obtain

$$G_\beta(x,y) = G_0(x,y) - c_1 k(\beta)|x-y|^{(2\alpha-d)\wedge 0} + E_\beta(x,y).$$

Here $k(\beta)$ is a function depending on d/α and $E_\beta(x,y)$ has smaller order than $k(\beta)$. Secondary, we consider the time changed process by μ for β-killed process of $\{X_t\}$ to obtain the representation of $G^\mu_\beta \mu$, since μ is not necessarily absolutely continuous. The Green operator of this process is given by $f \to \int_{\mathbb{R}^d} G_\beta(\cdot,y) f(y) \mu(dy)$, and a compact operator on $L^2(\mu)$. Thus, we can apply the perturbation theory and conclude that $k(\beta) G^\mu_\beta \mu$ converges $\mathscr{E}$-weakly as $\beta \to 0$. Since P^μ_ε admits Green-tight integral kernel, we can strengthen this convergence to pointwise one and obtain the following result:

Theorem 1. *(Takeda-W. 2014)*
Suppose $\{X_t\}$ is the transient, rotationally invariant α-stable process and μ is a critical measure with compact support. As $t \to \infty$, Feynman-Kac functional satisfies

$$\mathbb{E}_x[\exp(A^\mu_t)] \sim \begin{cases} c_1 h_0(x) t^{d/\alpha - 1} & (1 < d/\alpha < 2), \\ c_2 h_0(x) t/\log t & (d/\alpha = 2), \\ c_3 h_0(x) t & (d/\alpha > 2), \end{cases}$$

where $h_0(x)$ is the ground state of $\mathscr{E}^\mu$.

References

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