Identification of noncausal functions from the stochastic Fourier coefficients without the aid of a Brownian motion

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Let $f(t,\omega)$ be a random function on $[0,1]\times\Omega$ and $\{e_n(t)\}$ be a CONS in $L^2([0,1];\mathbb{C})$. The system $\{\int_0^1 f(t,\omega)\overline{e_n(t)}dW_t\}$ is called the stochastic Fourier coefficients (SFCs in abbr.) of $f(t,\omega)$. It is of course these stochastic integrals should be defined adequately. Let us consider whether $f(t,\omega)$ is identified from SFCs of $f(t,\omega)$.

Let $e_n(t)=\mathrm{e}^{2\pi int}$, $n\in\mathbb{Z}$. S.Ogawa [1], S.Ogawa and I [2,3] have studied this problem in the framework of the theory of the Skorokhod integral *with* the aid of a Brownian motion. Recently, however, S. Ogawa [4] obtained the affirmative answer *without* the aid of a Brownian motion from the stochastic Fourier transform if $f(t,\omega)$ is a nonnegative causal function. In this talk we will develop his method to the case where $f(t,\omega)$ is noncausal.

In this talk we assume the following conditions.

- $f(t,\omega)$ is differentiable with respect to t for almost all ω,
- $\displaystyle\int_0^1 f(t,\omega)dt\in L^2(\Omega,dP)$, $f'(t,\omega)\Big(=\frac{\partial}{\partial t}f(t,\omega)\Big)\in L^2([0,1]\times\Omega,dtdP)$,
- $e_n(t)=\mathrm{e}^{2\pi int},\quad n\in\mathbb{Z}$.

We define the stochastic Fourier coefficients through the Ogawa integral. We use the symbol d_*W_t for Ogawa integral. We remark that $f(t,\omega)$ under our conditions is Ogawa integrable and satisfies

$$\int_0^1 f(t,\omega)d_*W_t=f(1,\omega)W_1-\int_0^1 W_t f'(t,\omega)dt. \tag{1}$$

We denote the SFC $\int_0^1 f(t,\omega)\overline{e_n(t)}d_*W_t$ by $\tilde{f}_n$.

Proposition 1. *$\{\tilde{f}_n,n\in\mathbb{Z}\}$ is uniformly bounded in $L^1(dP)$.*

Since it holds that

$$\lim_{N.M\to\infty}E\left[\sup_{0\leqq t\leqq 1}\left|\sum_{n\neq 0,|n|\leqq N}\frac{1}{-4\pi^2n^2}\tilde{f}_ne_n(t)-\sum_{-4\pi^2n\neq 0,|n|\leqq M}\frac{1}{-4\pi^2n^2}\tilde{f}_ne_n(t)\right|\right]=0,$$

Proposition 2. *There exists $S(t)(=S(t,\omega))\in C([0,1])$ a.s. such that*

$$\lim_{N\to\infty}E\left[\sup_{0\leqq t\leqq 1}\left|\sum_{n\neq 0,|n|\leqq N}\frac{1}{-4\pi^2n^2}\tilde{f}_ne_n(t)-S(t)\right|\right]=0$$

We call $S(t)$ the $\{(-4\pi^2 n^2)^{-1}\}$-stochastic Fourier transform of $\{\hat{f}_n\}$.

From (1) and the integration by parts formula we have

$$\begin{aligned} S(t) &= -\frac{1}{2}\left(f(1,\omega)W_1 - \int_0^1 W_t f'(t,\omega)dt\right)\left(\frac{1}{6} - t + t^2\right) \\ &- \left(\int_0^1 \left(\int_0^t W_s f'(s,\omega)ds\right)dt - \int_0^1 W_t f(t,\omega)dt\right)\left(\frac{1}{2} - t\right) \\ &- \left(\int_0^t \int_0^s W_u f'(u,\omega)duds - \int_0^1 \int_0^t \int_0^s W_u f'(u,\omega)dudsdt\right) \\ &+ \left(\int_0^t W_s f(s,\omega)ds - \int_0^1 \int_0^t W_s f(s,\omega)dsdt\right) \end{aligned}$$

for all $t \in (0,1)$ almost surely. Since the right hand side above is differentiable in $t \in (0,1)$, so is $S(t)$ and we have

$$\begin{aligned} S'(t) &= -\frac{1}{2}\left(f(1,\omega)W_1 - \int_0^1 W_t f'(t,\omega)dt\right)(-1+2t) \\ &+ \left(\int_0^1 \left(\int_0^t W_s f'(s,\omega)ds\right)dt - \int_0^1 W_t f(t,\omega)dt\right) \\ &- \int_0^t W_u f'(u,\omega)du + W_t f(t,\omega) \end{aligned}$$

if $t \in (0,1)$. Thus it holds that for all fixed $s \in (0,1)$

$$\begin{aligned} &\limsup_{t\downarrow s} \frac{S'(t) - S'(s)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} \\ &= \limsup_{t\downarrow s} \frac{W_t f(t,\omega) - W_s f(s,\omega)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = f(s,\omega) \quad a.s. \end{aligned} \tag{2}$$

We note that the set on which (2) fails depends on s. Set $\mathbb{S} \subset (0,1)$ be a countable dense subset, then we have

Theorem 1.

$$\limsup_{t\downarrow s} \frac{S'(t) - S'(s)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = f(s,\omega)$$

for all $s \in \mathbb{S}$ almost surely. If $s \notin \mathbb{S}$, then $f(s,\omega) = \lim_{t\to s, t\in\mathbb{S}} f(t,\omega)$ holds.

References [1] S. Ogawa, *On a stochastic Fourier transformation*, Stochastics, 85-2 (2013), 286–294. [2] S. Ogawa and H. Uemura, *On a stochastic Fourier coefficient : case of noncausal functions*, J. Theoret. Probab. 27-2 (2014), 370 – 382. [3] S. Ogawa and H. Uemura, *Identification of noncausal Itô processes from the stochastic Fourier coefficients*, Bull. Sci. Math. 138-1 (2014), 147–163. [4] S. Ogawa, *A direct inversion formula for SFT*, to appear in Sankhya A, DOI 10.1007/s13171-014-0056-1.