A NOTE ON CONVERGENCE RATES FOR STABILITY PROBLEMS OF SDES

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Consider the following sequence of one-dimensional stochastic differential equations

$$X_n(t) = X_n(0) + \int_0^t b_n(X_n(s))ds + \int_0^t \sigma_n(X_n(s))dW_s, \tag{1}$$

and consider the solution X given by

$$X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW_s, \tag{2}$$

where $\{W_s\}_{s\geq 0}$ is a Wiener process and $b_n : \mathbb{R} \to \mathbb{R}$ and $\sigma_n : \mathbb{R} \to \mathbb{R}$ for $n \in \mathbb{N}$ are coefficients which tend to b and σ respectively in some sense, as $n \to \infty$. The convergence of the sequence $\{X_n\}_{n\in\mathbb{N}}$ to X named as stability problems was introduced by Stroock and Varadhan in [11] to solve the martingale problems for unbounded coefficients b and σ. On the other hand, the stability approach in the sense of *strong* to the continuous diffusion coefficients, typically Hölder continuous of exponent $(\frac{1}{2}+\gamma)$ for $0 \leq \gamma \leq 1/2$, has been further developed by Kawabata-Yamada [6]. The study of strong solution for irregular coefficients was treated by Nakao in [9] and also Zvonkin and Krylov in [7] where σ is bounded below by a positive constant and is of bounded variation on any compact interval. Then the result of [9] were extended to by Le Gall in [8], who proved the stability problems on the diffusion coefficients are positive and squared finite quadratic variation.

In another important development, the "rate" of convergence of the *Euler-Maruyama schemes* to the solution of (2) was drawn primarily from the paper Deelstra and Delbaen in [1] and their results has been considerably generalized by Gyöngy and Rásonyi in [3]: The convergence rate of *Euler-Maruyama schemes* with the diffusion coefficients satisfying $(\frac{1}{2}+\gamma)$-Hölder continuous and the suitable drift coefficients in L^1 is bounded by $n^{-\gamma}$ where $0 < \gamma \leq 1/2$, however, that is bounded by $(\log n)^{-1}$ in the case where the diffusion coefficients $(1/2)$-Hölder continuous, $\gamma = 0$.

The result suggests that the rate of convergence may depend also on the modulus continuity of diffusion coefficients. Therefore the goal of this research is to estimate the strong convergence rate of *stability problems*. To be more precious, let us consider the following drift-less stability problems,

$$X_n(t) - X_n(0) = \int_0^t \sigma_n(X_n(s))dW_s, \quad X(t) - X(0) = \int_0^t \sigma(X_n(s))dW_s,$$

for $t \geq 0$ and $X_n(0) \equiv X(0)$.

As a first result, the strong convergence rate is given of the stability problem when the coefficients satisfy the modulus continuity; σ and σ_n are $\left(\frac{1}{2}+\gamma\right)$-Hölder continuous and σ_n converges to σ uniformly. Then, there exists a positive constant C_1 such that

$$\mathbb{E}(|X(t) - X_n(t)|) \leq \begin{cases} C_1 n^{-\gamma} & (0 < \gamma \leq 1/2) \\ C_1 (\log n)^{-1} & (\gamma = 0). \end{cases}$$

Since the coefficient may be discontinuous under the following Nakao-Le Gall condition, a typically example is Skew Brownian motions [4], it seems to be very interesting to investigate the rate of convergence of the stability problems under the condition:

Definition 1 (Nakao-Le Gall condition)**.** *We say that a real valued function σ satisfies the Nakao-Le Gall condition and write $\sigma \in \mathcal{C}_{NL}(\epsilon, f)$ if σ satisfies the following statements:*

(*i*) *There exists a positive real number* ϵ *such that*

$$\epsilon \le \sigma(x)$$

holds for any x *in* $\mathbb{R}$.

(*ii*) *There exists a monotone increasing function* f *such that*

$$|\sigma(x) - \sigma(y)|^2 \le |f(x) - f(y)|$$

holds for every x *and* y *in* $\mathbb{R}$.

(*iii*) *In addition, the function* f *is bounded on* $\mathbb{R}$,

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)| < \infty.$$

In this presentation, the strong convergence rate of the Nakao-Le Gall condition will be given as the local time argument plays an important role to estimate. Thanks to the extended local time expression for rotation invariant and α-stable processes Z given by Fitzsimmons and Getoor [2], K. Yamada [12], and see also [10]. Then the result will be extended to the rotation invariant and index α process Z driven stochastic differential equations;

$$X_n(t) - X_n(0) = \int_0^t \sigma_n(X_n(s))dZ_s, \quad X(t) - X(0) = \int_0^t \sigma(X_n(s))dZ_s,$$

for $t \ge 0$ and $X_n(0) \equiv X(0)$.

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