Integration by parts formulas concerning maxima of some SDEs with applications

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Introduction

We shall firstly deal with the following one-dimensional stochastic differential equation (SDE),

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

and consider discrete time maximum and continuous time maximum which are defined by $M_T^n := \max\{X_{t_1}, \cdots, X_{t_n}\}$ and $M_T := \max_{0 \le t \le T} X_t$, respectively. Secondly, we will deal with the following multi-dimensional SDE,

$$Z_t^i = z_0^i + \int_0^t V_0^i(Z_s)ds + \sum_{j=1}^{d} \int_0^t V_j^i(Z_s) \circ dW_s^j,\ 1 \le i \le d,$$

and consider the random variable defined by $\hat{M}_T := \max\{Z_T^1, \cdots, Z_T^d\}$.

Introduction

- Our main goal is to prove integration by parts (IBP) formulas for M_T^n, M_T and $\hat{M}_T$.
  (i.e. $E[\varphi'(F)G] = E[\varphi(F)H(F,G)], \forall \varphi \in C^1$)
- IBP formulas are used to study the probability density functions by taking $G = 1$.
- In addition, IBP formulas are used to compute the risks of options in finance.

M_T^n, M_T and $\hat{M}_T$ are important random variables, especially in finance.
Indeed, for payoff functions f,

- $f(M_T), f(M_T^n)$: Lookback type option
- $f(\hat{M}_T)$: Rainbow type option

Outline of the talk

1. Previous works
2. Discrete time maximum M_T^n
3. Continuous time maximum M_T
4. Maximum of components $\hat{M}_T$

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Previous works

Study of density functions on maxima.

- Nualart, D., Vives, J.: Continuité absolue de la loi du maximum d'un processus continu. C. R. Acad. Sci. Paris Ser. 1 Math. **307**(7), 349-354 (1988). → A sufficient condition so that the law of continuous time maximum of a one-dimensional continuous process is absolutely continuous.
- Florit, C., Nualart, D.: A local criterion for smoothness of densities and application to the supremum of the Brownian sheet. Stat. Probab. Lett. **22**(1), 25-31 (1995). → The smoothness of the density function of the continuous time maximum of the Brownian sheet.
- Lanjri Zadi, N., Nualart, D.: Smoothness of the law of the supremum of the fractional Brownian motion. Electron. Comm. Probab. **8**, 102-111 (2003). → The smoothness of the density function of the continuous time maximum of the fractional Brownian motion.

Previous works

Study of density functions on maxima.

- Fournier, N., Printems, J.: Absolute continuity for some one-dimensional processes. Bernoulli **16**(2), 343-360 (2010).→Absolute continuity of the law of a solution to a one-dimensional SDE with coefficients depending on the continuous time maximum of the solution.
- Hayashi, M., Kohatsu-Higa, A.: Smoothness of the distribution of the supremum of a multi-dimensional diffusion process. Potential Anal. **38**(1), 57-77 (2013).→ The smoothness of the density function of the joint law of a multi-dimensional SDE at the time when a component attains its maximum.
- N.: Absolute continuity of the laws of a multi-dimensional stochastic differential equation with coefficients dependent on the maximum. Stat. Probab. Lett. **83**(11), 2499-2506 (2013).→ Multi-dimensional case of [Fournier, N., Printems, J.].

Previous works

Computation of Greeks (risks of options).

- Gobet, E., Kohatsu-Higa, A.: Computation of greeks for barrier and look-back options using Malliavin calculus, Electron. Comm. Probab. **8**, 51-62 (2003). $\rightarrow$ Computation of greeks (delta and gamma) of options depending on the continuous or discrete time maximum of a one-dimensional SDE.
- Bernis, G., Gobet, E., Kohatsu-Higa, A.: Monte Carlo evaluation of Greeks for multidimensional barrier and lookback options. Math. Finance **13**(1), 99-113 (2003). $\rightarrow$ Multi-dimensional Black-Scholes model case.
- N.: Volatility risk for options depending on extrema and its estimation using kernel methods. preprint (submitted). $\rightarrow$ Computation of greeks (vega) of options depending on continuous time maximum of a one-dimensional SDE.

Outline

1. Previous works
2. Discrete time maximum M_T^n
3. Continuous time maximum M_T
4. Maximum of components $\hat{M}_T$

Discrete time maximum

$(\Omega, \mathcal{F}, P)$: a complete probability space.
$\{W_t, t \in [0, \infty)\}$: one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$.
We consider the SDE:

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \qquad (1)$$

Fix $T > 0$ and a time partition $0 \leq t_1 < \cdots < t_n = T$.
Define $M_T^n := \max\{X_{t_1}, \cdots, X_{t_n}\}$ and
$M_T := \max_{0 \leq t \leq T} X_t$.

Discrete time maximum

Assumption (A)

(A1) For $t \in [0, \infty)$, $b(t, \cdot), \sigma(t, \cdot) \in C_b^2(\mathbb{R}; \mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t, \cdot)$ and $\sigma(t, \cdot)$ do not depend on t.

(A2) There exists $c > 0$ such that

$$|\sigma(t, x)| \geq c$$

holds, for any $x \in \mathbb{R}$ and $t \in [0, \infty)$.

Discrete time maximum

Theorem 1

Assume (A). Let $G \in \mathbb{D}^{1,\infty}$ *and* $t_1 > 0$. *Then there exists a random variable* $H^n_T(G)$ *such that* $H^n_T(G)$ *belongs to* $L^p(\Omega)$ *for any* $p \geq 1$, *and*

$$E^P\left[\varphi'(M^n_T)G\right] = E^P[\varphi(M^n_T)H^n_T(G)] \tag{2}$$

holds for any $\varphi \in C^1_b(\mathbb{R};\mathbb{R})$.

Remark 1

In the case that $t_1 = 0$, *(2) in Theorem 1 is valid for any* $\varphi \in C^1_b(\mathbb{R};\mathbb{R})$ *whose support is involved in* (x_0, ∞).

Proof

We define

$$u_r^n := \frac{Y_r}{\sigma(r,X_r)}\left[\frac{1}{t_1 Y_{t_1}}\mathbf{1}_{[0,t_1)}(r) + \sum_{k=2}^{n}\left(\frac{1}{Y_{t_k}} - \frac{1}{Y_{t_{k-1}}}\right)\frac{1}{t_k - t_{k-1}}\mathbf{1}_{[t_{k-1},t_k)}(r)\right],$$

for $r \in [0,T]$, where $Y_t := \frac{\partial X_t}{\partial x_0}$.
Define $A_1 := \{X_{t_1} = M_T^n\}$,$A_k := \{X_{t_1} \neq M_T^n, \cdots, X_{t_{k-1}} \neq M_T^n, X_{t_k} = M_T^n\}, k = 2, \cdots, n$. From the local property of the Malliavin derivative, we get

$$\begin{aligned}\int_0^T D_r(\varphi(M_T^n))u_r^n dr &= \varphi'(M_T^n)\sum_{k=1}^{n}\int_0^{t_k} D_r X_{t_k} u_r^n dr \mathbf{1}_{A_k} \\ &= \varphi'(M_T^n)\sum_{k=1}^{n} Y_{t_k}\int_0^{t_k}\frac{\sigma(r,X_r)}{Y_r}u_r^n dr \mathbf{1}_{A_k}.\end{aligned}$$

Proof

By the definition of u^n, one has

$$Y_{t_k} \int_0^{t_k} \frac{\sigma(r, X_r)}{Y_r} u_r^n dr = Y_{t_k} \left[\frac{1}{Y_{t_1}} + \sum_{l=2}^{k} \left(\frac{1}{Y_{t_l}} - \frac{1}{Y_{t_{l-1}}} \right) \right] = 1,$$

and

$$\int_0^T D_r(\varphi(M_T^n)) u_r^n dr = \varphi'(M_T^n) \sum_{k=1}^{n} \mathbf{1}_{A_k} = \varphi'(M_T^n).$$

Therefore, the duality gives

$$E^P[\varphi'(M_T^n)G] = E^P[\int_0^T D_r(\varphi(M_T^n)) G u_r^n dr] = E^P[\varphi(M_T^n)\delta(G u_\cdot^n)],$$

where δ denotes the Skorohod integral and finally (2) follows by taking $H_T^n(G) = \delta(G u_\cdot^n)$.

Proof

In the case that $t_1 = 0$, we define

$$\tilde{u}_r^n := \frac{Y_r}{\sigma(r, X_r)} \left[\frac{1}{t_2 Y_{t_2}} \mathbf{1}_{[0,t_2)}(r) + \sum_{k=3}^{n} \left(\frac{1}{Y_{t_k}} - \frac{1}{Y_{t_{k-1}}} \right) \frac{1}{t_{k+1} - t_k} \mathbf{1}_{[t_{k-1},t_k)}(r) \right],$$

for $r \in [0, T]$. If $supp \varphi \subset (x_0, \infty)$, then we have $\varphi'(M_T^n)\mathbf{1}_{A_1} = 0$. Thus, we get

$$\int_0^T D_r(\varphi(M_T^n))\tilde{u}_r^n dr = \varphi'(M_T^n) \sum_{k=2}^{n} \mathbf{1}_{A_k} + \varphi'(M_T^n)\mathbf{1}_{A_1} = \varphi'(M_T^n),$$

and (2) follows by taking $H_T^n(G) = \delta(G\tilde{u}_\cdot^n)$ □

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Continuous time maximum

In [Hayashi, M., Kohatsu-Higa, A.], the smoothness of the density function of $F := (X^1_\theta, \cdots, X^d_\theta)$ where $X = (X^1, \cdots, X^d)$ is a solution to a multi-dimensional SDE and $\theta := \arg\max_{t \le T} X^1_t$ is proved by means of the IBP formula.
The vector fields of the coefficients are assumed to be commutative and the explicit representation of the solution by using the exponential map of the vector fields is used to obtain the IBP formula.

Continuous time maximum

Assumption (A)'

The diffusion coefficient of (1) is of the form $\sigma(t,x) = \sigma_1(t)\sigma_2(x)$ and the following assumption.

(A1)' For $t \in [0,\infty)$, $b(t,\cdot) \in C_b^2(\mathbb{R};\mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t,\cdot)$ do not depend on t.

(A2)' $\sigma_1(\cdot) \in C_b^0([0,\infty);\mathbb{R})$ and there exists $c_1 > 0$ such that $|\sigma_1(t)| \geq c_1$ for any $t \in [0,\infty)$.

(A3)' $\sigma_2(\cdot) \in C_b^3(\mathbb{R};\mathbb{R}_+)$(or respectively $C_b^3(\mathbb{R};\mathbb{R}_-)$), $x \mapsto \sigma_2(x)$ is increasing (respectively decreasing) and there exists $c_2 > 0$ such that $|\sigma_2(x)| \geq c_2$ for any $x \in \mathbb{R}$.

Continuous time maximum

Theorem 2

Assume (A)'. Let $G \in \mathbb{D}^{1,\infty}$ *and* $a_0 > x_0$ *be fixed arbitrarily. Then there exists a random variable* $H_T(G, a_0)$ *such that* $H_T(G, a_0)$ *belongs to* $L^p(\Omega)$ *for any* $p \geq 1$*, and*

$$E^P[\varphi'(M_T)G] = E^P[\varphi(M_T)H_T(G, a_0)]$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$ *whose support is involved in* (a_0, ∞)*.*

Proof

Y_t is given by $Y_t = e^{\int_0^t (b' - \frac{(\sigma')^2}{2})(s,X_s)ds + \int_0^t \sigma'(s,X_s)dW_s}$.
We define $F(x) := \int_0^x \frac{\sigma_2'(y)}{\sigma_2(y)} dy$ for $x \in \mathbb{R}$ and $g(t,x) := (b' - \frac{\sigma\sigma''}{2} - \frac{b\sigma'}{\sigma})(t,x)$ for $(t,x) \in [0,\infty) \times \mathbb{R}$. By using Itô's formula, we get

$$\begin{aligned} e^{-F(x_0)+F(X_t)} &= e^{\int_0^t \sigma'(s,X_s)dW_s + \int_0^t (b' - \frac{(\sigma')^2}{2})(s,X_s)ds - \int_0^t g(s,X_s)ds} \\ &= Y_t e^{-\int_0^t g(s,X_s)ds}, \ t \in [0,T]. \end{aligned}$$

$x \mapsto F(x)$ is increasing, due to (A3)', thus,

$$\begin{aligned} \max_{0 \le t \le T} \{Y_t e^{-\int_0^t g(s,X_s)ds}\} &= \max_{0 \le t \le T} \{e^{-F(x_0)+F(X_t)}\} \\ &= e^{-F(x_0)+F(X_{\bar{\tau}_T})} = Y_{\bar{\tau}_T} e^{-\int_0^{\bar{\tau}_T} g(s,X_s)ds} \end{aligned}$$

follows, where $\bar{\tau}_T := \arg\max_{0 \le t \le T} X_t$.

We note that $\bar{\tau}_T$ is defined uniquely P-a.s. from Girsanov's theorem and a representation of a martingale w.r.t. a time changed Brownian motion.

Proof

We fix $a_0 > x_0$. Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a smooth function with bounded derivatives such that

$$\rho(x) = \begin{cases} 0, x > a_0 - x_0 \\ 1, x \leq \frac{a_0 - x_0}{2}, \end{cases}$$

holds. Let $\{\hat{Z}_t, t \in [0, T]\}$ be an adapted process satisfying the following conditions.

(H1) $\hat{Z}_0 = 0$ and $|X_t - x_0| \leq \hat{Z}_t$ for $t \in [0, T]$, P-a.s.

(H2) $[0, T] \ni t \mapsto \hat{Z}_t \in \mathbb{R}_+$ is non-decreasing.

(H3) $\int_0^T \rho(\hat{Z}_s)ds$ belongs to $\mathbb{D}^{1,2}$ and $E^P[\int_0^T |D_r(\int_0^T \rho(\hat{Z}_s)ds)|^p dr] < \infty$ holds for any $p \geq 1$.

(H4) There exist a positive function $a : \mathbb{N} \to \mathbb{R}_+$ and $\underline{q} \geq 1$ such that $\lim_{q\to\infty} a(q) = \infty$ holds and for any $q > \underline{q}$, one has $E^P[|\hat{Z}_t|^q] \leq c_q(T)t^{a(q)}, \forall t \in [0, T]$, where $c_q(T)$ is a positive constant.

Proof

We define

$$u_r := \frac{1}{\max\limits_{0\le t\le T}\left\{Y_t e^{-\int_0^t g(s,X_s)ds}\right\}} \frac{1}{\int_0^T \rho(\hat{Z}_s)ds}$$
$$\times \frac{Y_r}{\sigma(r,X_r)}\left[-g(r,X_r)e^{-\int_0^r g(s,X_s)ds}\int_0^r \rho(\hat{Z}_s)ds + e^{-\int_0^r g(s,X_s)ds}\rho(\hat{Z}_r)\right],$$

for $r \in [0,T]$. Note that

$$\left[-g(r,X_r)e^{-\int_0^r g(s,X_s)ds}\int_0^r \rho(\hat{Z}_s)ds + e^{-\int_0^r g(s,X_s)ds}\rho(\hat{Z}_r)\right]$$
$$= \frac{d}{dr}\left(e^{-\int_0^r g(s,X_s)ds}\int_0^r \rho(\hat{Z}_s)ds\right).$$

Proof

By the definition of u, we have

$$\int_0^T D_r M_T \cdot u_r dr = Y_{\bar{\tau}_T} \int_0^{\bar{\tau}_T} \frac{\sigma(r, X_r)}{Y_r} u_r dr$$

$$= \frac{1}{\max_{0 \le t \le T} \{ Y_t e^{-\int_0^t g(s,X_s)ds} \}} \frac{Y_{\bar{\tau}_T}}{\int_0^T \rho(\hat{Z}_s)ds} \int_0^{\bar{\tau}_T} \frac{d}{dr} \left(e^{-\int_0^r g(s,X_s)ds} \int_0^r \rho(\hat{Z}_s)ds \right) dr$$

$$= \frac{e^{\int_0^{\bar{\tau}_T} g(s,X_s)ds}}{\int_0^T \rho(\hat{Z}_s)ds} \int_0^{\bar{\tau}_T} \frac{d}{dr} \left(e^{-\int_0^r g(s,X_s)ds} \int_0^r \rho(\hat{Z}_s)ds \right) dr$$

$$= \frac{\int_0^{\bar{\tau}_T} \rho(\hat{Z}_s)ds}{\int_0^T \rho(\hat{Z}_s)ds}.$$

Proof

On the event $\{M_T > a_0\}$, one has $\inf\{t > 0; \hat{Z}_t \geq a_0 - x_0\} \leq \bar{\tau}_T$, thus, from the support condition of ρ, we have

$$\frac{\int_0^{\bar{\tau}_T} \rho(\hat{Z}_s)ds}{\int_0^T \rho(\hat{Z}_s)ds} = \frac{\int_0^{\bar{\tau}_T} \rho(\hat{Z}_s)ds}{\int_0^{\bar{\tau}_T} \rho(\hat{Z}_s)ds} = 1,$$

on the event $\{M_T > a_0\}$. Therefore, the result follows by taking

$H_T(G, a_0) = \delta(Gu.)$. □

Proof

A remark

Define $T_{a_0} := \inf\{t > 0; \hat{Z}_t \geq \frac{a_0 - x_0}{2}\} \wedge T$ and take $\varepsilon > 0$ such that $\varepsilon < T$, then we have, for $q > \underline{q}$,

$$P\left(\int_0^T \rho(\hat{Z}_s)ds \leq \varepsilon\right) \leq P\left(\int_0^{T_{a_0}} \rho(\hat{Z}_s)ds \leq \varepsilon\right) = P\left(T_{a_0} \leq \varepsilon\right)$$
$$\leq P\left(\hat{Z}_\varepsilon \geq \frac{a_0 - x_0}{2}\right) \leq 2^q \frac{E^P[\hat{Z}_\varepsilon^q]}{(a_0 - x_0)^q}$$
$$\leq \left(\frac{2}{a_0 - x_0}\right)^q c_q(T)\varepsilon^{a(q)}.$$

This implies $E^P[(\int_0^T \rho(\hat{Z}_s)ds)^{-p}] < \infty$ for $p \geq 1$, due to Fubini's theorem.

Proof

Example 1 Assume (A)'. Define

$$\hat{Z}_t := \max_{0\le s\le t}(X_s - x_0) - \min_{0\le s\le t}(X_s - x_0), \tag{3}$$

for $t \in [0, T]$. Then $\hat{Z}$ satisfies (H1)-(H4).

Example 2 Assume (A)'. Let γ be an even integer and m be a real number satisfying $0 < m < \frac{\gamma}{2} - 2$ and define

$$A_t := 4\int_0^t \int_0^t \frac{|X_s - X_u|^\gamma}{|s-u|^{m+2}} dsdu,$$

and

$$\hat{Z}_t := \frac{8(m+2)}{m} A_t^{\frac{1}{\gamma}} t^{\frac{m}{\gamma}} \tag{4}$$

for $t \in [0, T]$. Then $\hat{Z}$ satisfies (H1)-(H4) due to Garsia-Rodemich-Rumsey's lemma.

Expressions of the density functions

Proposition 1

Assume (A). Let $t_1 > 0$. Then the density function of M_T^n is given by

$$p_{M_T^n}(x) = E^P\left[\mathbf{1}_{\{M_T^n > x\}} H_T^n(1)\right], \tag{5}$$

for every $x \in \mathbb{R}$, where $H_T^n(1)$ is defined by $\delta(u_\cdot^n)$.
Assume (A)'. Let $a_0 > x_0$ be fixed arbitrarily. Then the density function of M_T is given by

$$p_{M_T}(x) = E^P\left[\mathbf{1}_{\{M_T > x\}} H_T(1, a_0)\right], \tag{6}$$

for every $x > a_0 > x_0$, where $H_T(1, a_0)$ is defined by $\delta(u_\cdot)$.

Expressions of the density functions

Remark 2

In the case that $t_1 = 0$*, under assumption (A), (5) is valid for* $x > x_0$ *by defining* $H_T^n(1) := \delta(\tilde{u}_\cdot^n)$*.*

Remark 3

Since $E^P[\delta(v)] = 0$ *holds for any* $v \in Dom\,\delta$*, we can express (5) and (6) as*

$$p_{M_T^n}(x) = -E^P\left[\mathbf{1}_{\{M_T^n < x\}} H_T^n(1)\right], \tag{7}$$

every $x \in \mathbb{R}$*, and*

$$p_{M_T}(x) = -E^P\left[\mathbf{1}_{\{M_T < x\}} H_T(1, a_0)\right],$$

for every $x > a_0 > x_0$*.*

Upper bounds of the density functions

For simplicity, we assume that each time partition is uniform, and $T \leq 1$.
$C(\sigma)$: a constant which bounds σ.

Theorem 3

Assume (A) and that $T \leq 1$. Moreover, we assume that the time partition is defined by $t_k = \frac{T}{n}k, 1 \leq k \leq n$(if $t_1 > 0$) or $t_k = \frac{T}{n-1}(k-1), 1 \leq k \leq n+1$ (if $t_1 = 0$). Then for any $p_1, p_2 > 1$, there exists $C(n, p_1, p_2) > 0$ such that

$$p_{M_T^n}(x) \leq C(n, p_1, p_2)\left(T^{\frac{1}{2p_1p_2}} + T^{\frac{1}{2p_1p_2}-\frac{1}{2}}\right) \times \left(\frac{1}{x - x_0 + \sqrt{(x-x_0)^2 + 2C(\sigma)^2 T}}\right)^{\frac{1}{p_1p_2}} e^{-\frac{1}{2p_1p_2}\frac{(x-x_0)^2}{C(\sigma)^2 T}},$$

holds for any $x > x_0$, where $C(n, p_1, p_2)$ does not depend on T.

Upper bounds of the density functions

Theorem 4

Assume (A)' and that $T \leq 1$. Moreover, we assume that $\hat{Z}$ is defined by (3) or (4). Then for any $p_1, p_2 > 1$, there exists $C(a_0, p_1, p_2) > 0$ such that

$$p_{M_T}(x) \leq C(a_0, p_1, p_2) \left(T^{\frac{1}{2p_1 p_2}} + T^{\frac{1}{2p_1 p_2} - \frac{1}{2}} \right) \times \left(\frac{1}{x - x_0 + \sqrt{(x - x_0)^2 + 2C(\sigma)^2 T}} \right)^{\frac{1}{p_1 p_2}} e^{-\frac{1}{2p_1 p_2} \frac{(x-x_0)^2}{C(\sigma)^2 T}},$$

holds for any $x > a_0 > x_0$, where $C(a_0, p_1, p_2)$ does not depend on T.

Proof

Define a probability measure Q by $\frac{dQ}{dP}\Big|_{\mathcal{F}_T} := e^{-\int_0^T \frac{b}{\sigma}(s,X_s)dW_s - \frac{1}{2}\int_0^T (\frac{b}{\sigma})^2(s,X_s)ds}$, then $\tilde{W}_t := W_t + \int_0^t \frac{b}{\sigma}(s,X_s)ds,\ t \in [0,T]$ is a one-dimensional Q-Brownian motion and X satisfies

$$X_t = x_0 + \int_0^t \sigma(s,X_s)d\tilde{W}_s,\ t \in [0,T].$$

Since $\langle X \rangle_t \geq ct$ (under (A)', we define $c := c_1 c_2$) holds for $t \in [0,\infty)$, X can be described as

$$X_t = x_0 + \hat{W}_{\langle X \rangle_t},\ t \in [0,\infty) \tag{8}$$

where $\hat{W} = \{\hat{W}_t, t \in [0,\infty)\}$ denotes a one-dimensional Q-Brownian motion.

Proof

By using (8) and $\langle X \rangle_T = \int_0^T \sigma^2(s, X_s) ds \le C(\sigma)^2 T$, we get

$$\max_{0 \le t \le T} X_t = x_0 + \max_{0 \le t \le T} \hat{W}_{\langle X \rangle_t} \le x_0 + \max_{0 \le t \le C(\sigma)^2 T} \hat{W}_t,$$

and

$$P(M_T > x) = E^Q \left[\mathbf{1}_{\{M_T > x\}} \frac{dP}{dQ} \Big|_{\mathcal{F}_T} \right]$$
$$\le Q \left(x_0 + \max_{0 \le t \le C(\sigma)^2 T} \hat{W}_t > x \right)^{\frac{1}{p}} E^Q \left[\left(\frac{dP}{dQ} \Big|_{\mathcal{F}_T} \right)^q \right]^{\frac{1}{q}},$$

for $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Proof

From the explicit density function for $\max_{0\le t\le C(\sigma)^2 T}\hat{W}_t$ and Komatsu's inequality, we have

$$Q\left(x_0+\max_{0\le t\le C(\sigma)^2 T}\hat{W}_t > x\right) = 2\int_{x-x_0}^{\infty}\frac{1}{\sqrt{2\pi C(\sigma)^2 T}}e^{-\frac{y^2}{2C(\sigma)^2 T}}dy$$

$$\le \frac{2\sqrt{2}}{\sqrt{\pi}}\left(\frac{x-x_0}{C(\sigma)\sqrt{T}}+\sqrt{\frac{(x-x_0)^2}{C(\sigma)^2 T}+2}\right)^{-1}e^{-\frac{1}{2}\frac{(x-x_0)^2}{C(\sigma)^2 T}}.$$

One has

$$E^P\left[|H_T^n(1)|^{q_1}\right]^{\frac{1}{q_1}} \le c_p(1+T^{-\frac{q_1}{2}})^{\frac{1}{q_1}} \le c_p(1+T^{-\frac{1}{2}}),$$

in the case that $t_k=\frac{T}{n}k, 1\le k\le n$ or $t_k=\frac{T}{n-1}(k-1), 1\le k\le n+1$.

Proof

Due to (5) and Hölder's inequality, one has

$$p_{M_T^n}(x) \leq E^P\left[\mathbf{1}_{\{M_T^n > x\}}\right]^{\frac{1}{p_1}} E^P\left[|H_T^n(1)|^{q_1}\right]^{\frac{1}{q_1}} = P\left(M_T^n > x\right)^{\frac{1}{p_1}} E^P\left[|H_T^n(1)|^{q_1}\right]^{\frac{1}{q_1}}$$

$$\leq Q\left(M_T^n > x\right)^{\frac{1}{p_1 p_2}} E^Q\left[\left(\left.\frac{dP}{dQ}\right|_{\mathcal{F}_T}\right)^{q_2}\right]^{\frac{1}{p_1 q_2}} E^P\left[|H_T^n(1)|^{q_1}\right]^{\frac{1}{q_1}},$$

for $p_i, q_i > 1$ satisfying $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $i = 1, 2$.

□

Upper bounds of the density functions

In the case that $t_1 > 0$, for $x < x_0$, the value of $p_{M_T^n}(x)$ does not vanish, and we use the expression (7) in order to obtain an upper bound of $p_{M_T^n}(x)$ for $x < x_0$.

Proposition 2

Assume (A) and that $T \leq 1$. Moreover, we assume that the time partition is defined by $t_k = \frac{T}{n}k, 1 \leq k \leq n$. Then for any $p_1, p_2 > 1$, there exists $C(n, p_1, p_2) > 0$ such that

$$p_{M_T^n}(x) \leq C(n, p_1, p_2)\left(T^{\frac{1}{2p_1p_2}} + T^{\frac{1}{2p_1p_2} - \frac{1}{2}}\right) \times \left(\frac{1}{x_0 - x + \sqrt{(x - x_0)^2 + 2C(\sigma)^2 T}}\right)^{\frac{1}{p_1p_2}} e^{-\frac{1}{2p_1p_2}\frac{(x-x_0)^2}{C(\sigma)^2 T}},$$

for $x < x_0$, where $C(n, p_1, p_2)$ does not depend on T.

Outline

Maximum of components

$(\Omega', \mathcal{F}', P')$: a complete probability space.
$\{W_t = (W_t^1, \cdots, W_t^d), t \in [0, \infty)\}$: d-dimensional Brownian motion on $(\Omega', \mathcal{F}', P')$.
First, we will deal with the following multi-dimensional SDE,

$$Z_t^i = z_0^i + \sum_{j=1}^{d} \int_0^t V_j^i(Z_s) \circ dW_s^j,\ 1 \leq i \leq d,$$

and consider the random variable defined by
$M_T^* := \max\{Z_T^1, \cdots, Z_T^d\}$.

Maximum of components

Assumption (B)

(B1) For each $1 \leq i, j \leq d$, $V_j^i(\cdot) \in C_b^2(\mathbb{R}^d; \mathbb{R})$.

(B2) There exists $c > 0$ such that

$$\langle \xi, a(x)\xi \rangle \geq c|\xi|^2,$$

holds for any $x, \xi \in \mathbb{R}^d$, where $a(x) := VV^T(x)$.

(B3) Vector fields $V_1, \cdots, V_d$ are commutative, that is

$$[V_i, V_j](x) = [V_j, V_i](x), 1 \leq i, j \leq d$$

hold for any $x \in \mathbb{R}^d$, where we have defined the Lie bracket by $[V_i, V_j](x) := \nabla V_j V_i(x) - \nabla V_i V_j(x)$.

(B4) For each $1 \leq i, j \leq d$, $(V^{-1})_j^i(\cdot) \in C_b^1(\mathbb{R}^d; \mathbb{R})$.

Maximum of components

Theorem 5

Assume (B). Let $G \in \mathbb{D}^{1,\infty}$. Then there exists a random variable $H_T^*(G)$ such that $H_T^*(G)$ belongs to $L^p(\Omega)$ for any $p \geq 1$, and

$$E^{P'}[\varphi'(M_T^*)G] = E^{P'}[\varphi(M_T^*)H_T^*(G)] \tag{9}$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$.

Proof

We define the set of events $A_1^* := \{X_T^1 = M_T^*\}$ and $A_k^* := \{X_T^1 \neq M_T^*, \cdots, X_T^{k-1} \neq M_T^*, X_T^k = M_T^*\}$ for $k = 2, \cdots, d$. Then by using the explicit representation with exponential maps, we get $M_T^* \in \mathbb{D}^{1,\infty}$ and

$$D_r^j M_T^* = \mathbf{1}_{[0,T]}(r) \sum_{i=1}^{d} V_j^i(X_T) \mathbf{1}_{A_i^*},$$

for $1 \leq j \leq d$. Let $\{h_r^*, r \in [0, T]\}$ be a one-dimensional process such that $\int_0^T h_r^* dr < \infty$, P'-a.s. Define $F_j := \sum_{k=1}^d (V^{-1})_k^j (X_T)$, $u_r^{*,j} := F_j h_r^*$. Moreover, we define a d-dimensional process by

$$u_r^* := (u_r^{*,1}, \cdots, u_r^{*,d})$$

for $r \in [0, T]$.

Proof

We easily get

$$\int_0^T \sum_{j=1}^d D_r^j(\varphi(M_T^*)) \cdot u_r^{*,j} dr = \varphi'(M_T^*) \int_0^T h_r^* dr.$$

Therefore, for $G \in \mathbb{D}^{1,\infty}$, one has

$$\begin{aligned} E^{P'}[\varphi'(M_T^*)G] &= E^{P'}\left[\int_0^T \sum_{j=1}^d D_r^j(\varphi(M_T^*)) \cdot \frac{Gu_r^{*,j}}{\int_0^T h_v^* dv} dr\right] \\ &= E^{P'}\left[\varphi(M_T^*)\delta\left(\frac{Gu_\cdot^*}{\int_0^T h_v^* dv}\right)\right], \end{aligned}$$

and (9) follows with $H_T^*(G) = \delta(\frac{Gu_\cdot^*}{\int_0^T h_v^* dv})$, as long as $\frac{u_\cdot^*}{\int_0^T h_v^* dv} \in Dom\, \delta$. □

Maximum of components

Consider the following multi-dimensional SDE,

$$Z_t^i = z_0^i + \int_0^t V_0^i(Z_s)ds + \sum_{j=1}^{d} \int_0^t V_j^i(Z_s) \circ dW_s^j, \ 1 \leq i \leq d$$

and consider the random variable defined by
$\hat{M}_T := \max\{Z_T^1, \cdots, Z_T^d\}$.

Assumption (B)'

In addition to (B), we assume

(B5) For each $1 \leq i \leq d$, $V_0^i(\cdot) \in C_b^1(\mathbb{R}^d; \mathbb{R})$.

Maximum of components

Theorem 6

Assume (B)'. Then there exists a random variable $\hat{H}_T$ such that $\hat{H}_T$ belongs to $L^p(\Omega)$ for any $p \geq 1$, and

$$E^{P'}[\varphi'(\hat{M}_T)] = E^{P'}[\varphi(\hat{M}_T)\hat{H}_T]$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$.

The proof is done from the Girsanov's theorem.

Upper bound of the density function

Proposition 3

Assume (B)' and that $T \le 1$. Then for any $p_1, p_2 > 1$, there exists $C(p_1, p_2) > 0$ such that

$$p_{\hat{M}_T}(x) \le C(p_1,p_2)\left(T^{\frac{1}{2p_1p_2}} + T^{\frac{1}{2p_1p_2}-\frac{1}{2}}\right) \times \sum_{i=1}^{d}\left(\frac{1}{x - z_0^i + \sqrt{(x-z_0^i)^2 + 2C_i^2 T}}\right)^{\frac{1}{p_1p_2}} e^{-\frac{1}{2p_1p_2}\frac{(x-z_0^i)^2}{C_i^2 T}},$$

holds for $x > \max\{z_0^1, \cdots, z_0^d\}$, where $C(p_1,p_2)$ does not depend on T.

Upper bound of the density function

Proposition 4

Assume (B)' and that $T \leq 1$. Then for any $p_1, p_2 > 1$, there exists $C(p_1, p_2) > 0$ such that

$$p_{\hat{M}_T}(x) \leq C(p_1,p_2)\left(T^{\frac{1}{2p_1p_2}} + T^{\frac{1}{2p_1p_2}-\frac{1}{2}}\right) \times \left(\frac{1}{z_0^i - x + \sqrt{(x-z_0^i)^2 + 2C_i^2 T}}\right)^{\frac{1}{p_1p_2}} e^{-\frac{1}{2p_1p_2}\frac{(z_0^i-x)^2}{C_i^2 T}},$$

holds for $x < \min\{z_0^1, \cdots, z_0^d\}$ and for $1 \leq i \leq d$, where $C(p_1, p_2)$ does not depend on T.

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