Integration by parts formulas concerning maxima of some SDEs with applications

Tomonori Nakatsu (Ritsumeikan University)

1 Introduction

In this talk, firstly, we shall deal with the following one-dimensional stochastic differential equation (SDE),

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{1}$$

where $b, \sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions and $\{W_t, t \in [0,\infty)\}$ denotes a one-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. We will consider discrete time maximum and continuous time maximum which are defined by $M_T^n := \max\{X_{t_1}, \cdots, X_{t_n}\}$ and $M_T := \max_{0\le t\le T} X_t$, respectively, where the time interval $[0,T]$ and the time partition $0 \le t_1 < \cdots < t_n = T$, $n \ge 2$ are fixed.

Secondly, we will deal with the following d-dimensional SDE,

$$Z_t^i = z_0^i + \int_0^t V_0^i(Z_s)ds + \sum_{j=1}^d \int_0^t V_j^i(Z_s) \circ dW_s^j,\ 1 \le i \le d,$$

where $V_j^i : \mathbb{R}^d \to \mathbb{R}$, $0 \le j \le d, 1 \le i \le d$ are measurable functions and $\circ dW^j$ denotes the Stratonovich integral with respect to a d-dimensional standard Brownian motion $\{W_t = (W_t^1, \cdots, W_t^d), t \in [0,\infty)\}$ defined on a probability space $(\Omega', \mathcal{F}', P')$. For this d-dimensional SDE, we shall consider the random variable defined by $\hat{M}_T := \max\{Z_T^1, \cdots, Z_T^d\}$, where $T > 0$ is fixed.

In this talk, we say that an integration by parts (IBP) formula for random variables F and G holds if there exists a random variable $H(F;G)$ such that $E^P[\varphi'(F)G] = E^P[\varphi(F)H(F;G)]$ holds for any φ in a class of C^1 functions, where $E^P[\cdot]$ denotes the expectation with respect to a probability measure P. The IBP formula is usually used to obtain expressions and upper bounds of the probability density function of F by taking $G = 1$. Meanwhile, in finance, IBP formulas play an important role in order to compute the risks of financial products, called greeks (see [1], for example).

Our goal is to prove IBP formulas for M_T^n, M_T and $\hat{M}_T$, in addition, to obtain the expressions and upper bounds of their probability density functions by means of the IBP formulas.

2 Main results

Assumption (A)

(A1) For $t \in [0,\infty)$, $b(t,\cdot), \sigma(t,\cdot) \in C_b^2(\mathbb{R};\mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t,\cdot)$ and $\sigma(t,\cdot)$ do not depend on t.

(A2) There exists $c > 0$ such that

$$|\sigma(t,x)| \ge c$$

holds, for any $x \in \mathbb{R}$ and $t \in [0,\infty)$.

Theorem 1. *Assume (A). Let $G \in \mathbb{D}^{1,\infty}$ and assume $t_1 > 0$. Then there exists a random variable $H_T^n(G)$ such that $H_T^n(G)$ belongs to $L^p(\Omega, \mathcal{F}, P)$ for any $p \ge 1$, and*

$$E^P[\varphi'(M_T^n)G] = E^P[\varphi(M_T^n)H_T^n(G)] \tag{2}$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$.

Remark 1. *In the case that $t_1 = 0$, (2) in Theorem 1 is valid for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$ whose support is included in (x_0,∞).*

Assumption (A)'

We assume that the diffusion coefficient of (1) is of the form $\sigma(t,x) = \sigma_1(t)\sigma_2(x)$ and the following assumption.

(A1)' For $t \in [0,\infty)$, $b(t,\cdot) \in C_b^2(\mathbb{R};\mathbb{R})$. Furthermore, all constants which bound the derivatives of $b(t,\cdot)$ do not depend on t.

(A2)' $\sigma_1(\cdot) \in C_b^0([0,\infty);\mathbb{R})$ and there exists $c_1 > 0$ such that $|\sigma_1(t)| \geq c_1$ for any $t \in [0,\infty)$.

(A3)' $\sigma_2(\cdot) \in C_b^3(\mathbb{R};\mathbb{R}_+)$(respectively, $C_b^3(\mathbb{R};\mathbb{R}_-)$), $x \mapsto \sigma_2(x)$ is increasing (respectively, decreasing) and there exists $c_2 > 0$ such that $|\sigma_2(x)| \geq c_2$ for any $x \in \mathbb{R}$.

Theorem 2. *Assume (A)'. Let $G \in \mathbb{D}^{1,\infty}$ and $a_0 > x_0$ be fixed arbitrarily. Then there exists a random variable $H_T(G,a_0)$ such that $H_T(G,a_0)$ belongs to $L^p(\Omega,\mathcal{F},P)$ for any $p \geq 1$, and*

$$E^P\left[\varphi'(M_T)G\right] = E^P\left[\varphi(M_T)H_T(G,a_0)\right]$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$ whose support is included in (a_0,∞).

Define

$$a(x) := VV^T(x),$$

for $x \in \mathbb{R}^d$, where V^T is the transpose matrix for V.

Assumption (B)

(B1) For each $1 \leq i,j \leq d$, $V_j^i(\cdot) \in C_b^2(\mathbb{R}^d;\mathbb{R})$.

(B2) There exists $c > 0$ such that

$$\langle \xi, a(x)\xi \rangle \geq c|\xi|^2,$$

holds for any $x, \xi \in \mathbb{R}^d$.

(B3) Vector fields $V_1,\cdots,V_d$ are commutative, that is

$$[V_i,V_j](x) = [V_j,V_i](x), 1 \leq i,j \leq d$$

hold for any $x \in \mathbb{R}^d$, where we have defined the Lie bracket by $[V_i,V_j](x) := \nabla V_j V_i(x) - \nabla V_i V_j(x)$.

(B4) For each $1 \leq i,j \leq d$, $(V^{-1})_j^i(\cdot) \in C_b^1(\mathbb{R}^d;\mathbb{R})$.

(B5) For each $1 \leq i \leq d$, $V_0^i(\cdot) \in C_b^1(\mathbb{R}^d;\mathbb{R})$.

Theorem 3. *Assume (B). Then there exists a random variable $\hat{H}_T$ such that $\hat{H}_T$ belongs to $L^p(\Omega',\mathcal{F}',P')$ for any $p \geq 1$, and*

$$E^{P'}[\varphi'(\hat{M}_T)] = E^{P'}[\varphi(\hat{M}_T)\hat{H}_T]$$

holds for any $\varphi \in C_b^1(\mathbb{R};\mathbb{R})$.

References

[1] Gobet, E., Kohatsu-Higa, A.: Computation of greeks for barrier and look-back options using Malliavin calculus. Electron. Commun. Probab. **8**, 51-62 (2003).

[2] Hayashi, M., Kohatsu-Higa, A.: Smoothness of the distribution of the supremum of a multi-dimensional diffusion process. Potential Anal. **38** (1), 57-77 (2013).

[3] Nakatsu, T.: Integration by parts formulas concerning maxima of some SDEs with applications to the study of density functions. Preprint.

[4] Nualart, D.: The Malliavin Calculus and Related Topics, 2nd edn. Probability and its Applications (New York), Springer-Verlag, Berlin (2006).