Stochastic heat equation arising from a certain branching systems in random environment

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Intorduction

In this talk, we will consider a stochastic heat equation:

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sigma(X_t(x))\dot{W}(t,x), \quad \sigma(0) = 0,$$
$$\lim_{t \to 0} X_t(x)dx = X_0(dx) \tag{1}$$

where W is a time-space white noise.
We construct a solution from a branching system in random environment for the case $\sigma(u) = \sqrt{u + 2u^2}$.

History

- Existence of solutions
  1. $\sigma(u)$ is Lipschitz and X_0 has a "continuous density". $\Rightarrow$ Existence and uniqueness of strong solution. (Cabana '70, Walsh '70s, Funaki '83, Iwata '87)
  2. $\sigma(u) = \sqrt{\gamma u}$ and X_0 is finite measure $\Rightarrow$ Existence and uniqueness of nonnegative weak solutions via "super-Brownian motion" (Konno-Shiga '88, Reimers '89).
  3. $|\sigma(u)| \leq C(1 + |u|)$ and X_0 has a "continuous density" $\Rightarrow$ Existence of weak solutions (Shiga '94).

History (cont'd)

- Uniqueness of solutions
  1. $\sigma(u) = u^\gamma$ ($\frac{1}{2} < \gamma < 1$) $\Rightarrow$ Uniqueness of nonnegative weak solution (Mytnik '99).
  2. $\sigma^2(u)$ is analytic and X is bounded $\Rightarrow$ Uniqueness of weak solution (Athreya-Tribe '00).
  3. $\sigma(u) = |u|^\gamma$ ($\frac{3}{4} < \gamma < 1$) $\Rightarrow$ Pathwise uniqueness (Mytnik-Perkins '11).
  4. $\sigma(u) = |u|^\gamma$ ($\frac{1}{2} \leq \gamma < \frac{3}{4}$) $\Rightarrow$ Pathwise nonuniqueness (Mueller-Mytnik-Perkins'14).

Super-Brownian motion

The stochastic heat equations (1) appear as the scaling limit process of some models.

For example,

- Cole-Hopf solution to KPZ $\Rightarrow \sigma(u) = u$. (Bertini-Giacomin '97)
- Stepping-Stone model $\Rightarrow \sigma(u) = \sqrt{u - u^2}$. (Shiga '88)
- Super-Brownian motion $\Rightarrow \sigma(u) = \sqrt{\gamma u}$.
- Long-range contact process (or voter model) $\Rightarrow \sigma(u) = \sqrt{u}$ and drift term $((cu - u^2)dt)$. (Mueller-Tribe '95)

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We focus on super-Brownian motion which appears as a scaling limit of critical branching random walks.

Branching diffusion

We start a simple model "Galton-Watson process" before considering super-Brownian motion.

Galton-Watson process

Let $x \geq 0$ and $N \in \mathbb{N}$. We call the following particle systems Galton-Watson process:

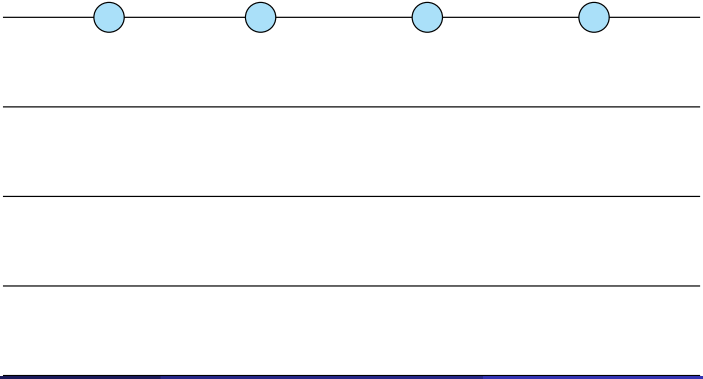
1. There exist $\lfloor xN \rfloor$-partcles at time 0.
2. Each particle independently reproduces two particles with probability p or vanishes with probability $1-p$ for each time,

where $0 < p < 1$.

Remark: We consider binary type for simplicity.

Example of Galton-Watson process

$$x = 1,\ N = 4 \text{ and } p = \frac{1}{2}.$$



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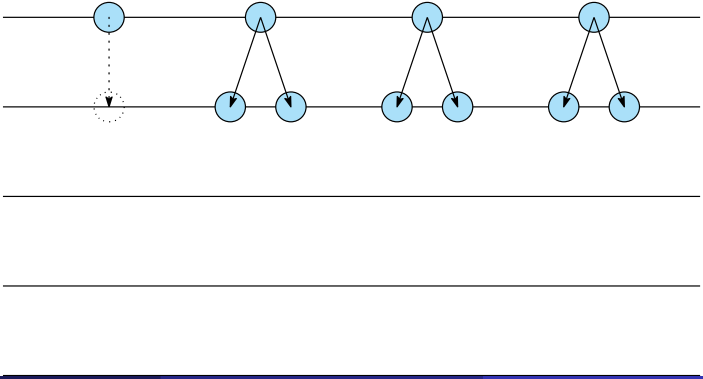

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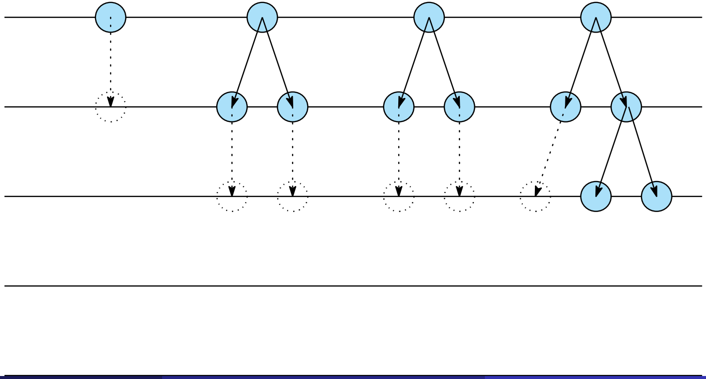

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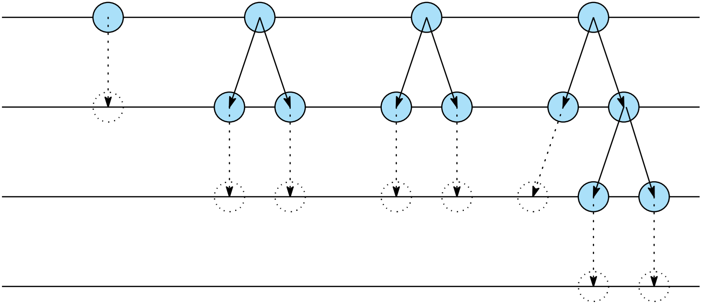


Branching diffusion

We say that Galton-Watson process is critical (supercritical, subcritical) when $p = \frac{1}{2}$ ($p > \frac{1}{2}$, $p < \frac{1}{2}$).
Let $B_n^{(N)}$ be the number of particles at time n. Then, we have the following theorem:

Theorem A (Feller '39, '51)

Let $X_t^{(N)} = \frac{1}{N} B_{\lfloor tN \rfloor}^{(N)}$ and $p = p^{(N)} = \frac{1}{2} + \frac{r}{2N}$ $(r \in \mathbb{R})$. Then,

$$X_\cdot^{(N)} \Rightarrow^{\exists} X_\cdot, \text{ in } D([0,\infty),\infty).$$

In particular, X is the strong unique solution of SDE:

$$dX_t = rX_t dt + \sqrt{X_t} dB_t, \quad X_0 = x,$$

where B_t is a one-dimensional Brownian motion.

Remark: The strong uniqueness holds by Yamada-Watanabe theorem.

Super-Brownian motion

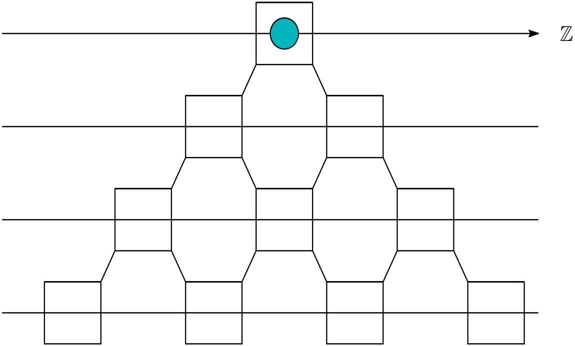
Next, we consider branching process with spatial motion on $\mathbb{Z}^d$.

Branching random walks

1. There exists particles at $x_1, \cdots, x_{M_N} \in \mathbb{Z}^d$ at time 0.
2. Each particle independently chooses nearest neighbor site with probability $\frac{1}{2d}$ and moves there.
3. Then, it is independently replaced by two particles with probability $\frac{1}{2}$ or erased with probability $\frac{1}{2}$.

Example of branching random walks

$$d = 1,\ M_N = 1,\ x_1 = 0.$$



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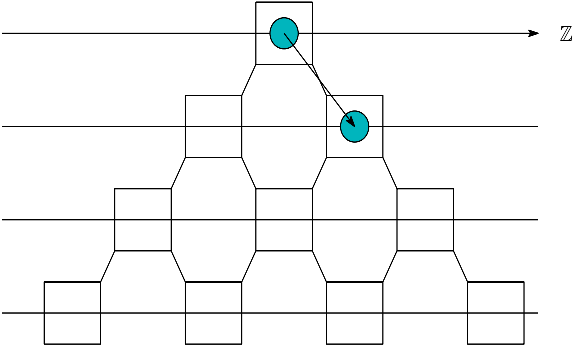

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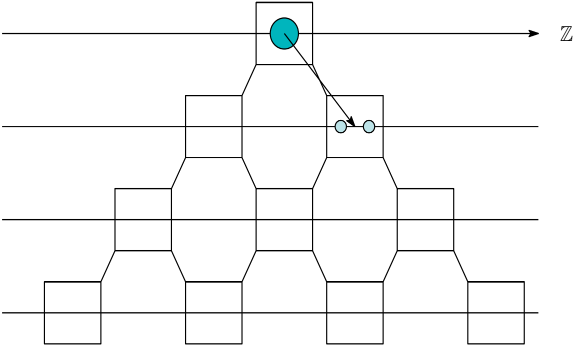

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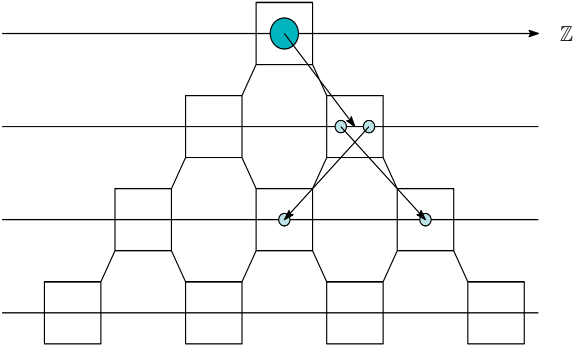

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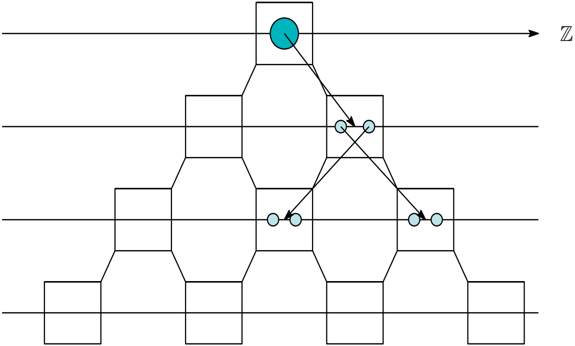

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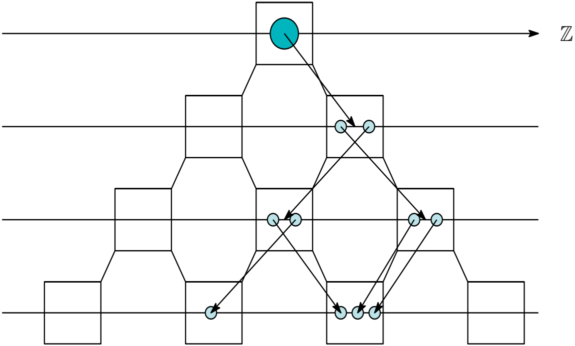

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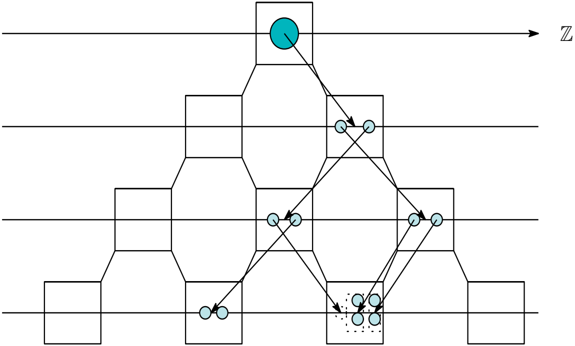


Super-Brownian motion

We define branching random walks as $\mathcal{M}_F(\mathbb{R}^d)$-valued process as follows:

$$X_0^{(N)}(dx) = \frac{1}{N}\sum_{i=1}^{M_N} \delta_{x_i/N^{1/2}}(dx)$$
$$X_t^{(N)}(A) = \frac{1}{N}\sharp\{\text{Particles in } \sqrt{N}A \text{ at time } \lfloor tN \rfloor\},$$

where $\mathcal{M}_F(\mathbb{R}^d)$ is the set of finite measure with the topology of weak convergence and $A \in \mathcal{B}(\mathbb{R}^d)$.

Super-Brownian motion

Theorem B (Watanabe '68, Dawson '75)

Suppose $X_0^{(N)} \Rightarrow X_0$ in $\mathcal{M}_F(\mathbb{R}^d)$. Then, we have that

$$X_\cdot^{(N)} \Rightarrow^{\exists} X_\cdot, \quad \text{in } D([0,\infty),\mathcal{M}_F(\mathbb{R}^d)).$$

In particular, X is the unique solution to the martingale problem:

$$\begin{cases} \text{For any } \phi \in C_b^2(\mathbb{R}^d) \\ Z_t(\phi) = X_t(\phi) - X_0(\phi) - \displaystyle\int_0^t X_s\left(\frac{1}{2d}\Delta\phi\right) ds \\ \text{is an } L^2\text{-continuous } \mathcal{F}_t^X\text{-martingale and} \\ \langle Z(\phi)\rangle_t = \displaystyle\int_0^t X_s(\phi^2) ds. \end{cases}$$

We call measure-valued process X a super-Brownian motion.

Super-Brownian motion

Super-Brownian motion is characterized by the solution of PDE:

Characterization via nonlinear-PDE

For $\phi \in C_b^{2,+}(\mathbb{R}^d)$, we define

$$E[\exp(-X_t(\phi))] = \exp(-X_0(u_t)).$$

Then, u is the unique solution to

$$\frac{\partial u}{\partial t} = \frac{1}{2d}\Delta u(x) - \frac{1}{2}u^2, \quad u(0,x) = \phi(x).$$

We remark that we can construct other super-Brownian motion characterized by

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u(x) + \beta u - \alpha u^p, \quad u(0,x) = \phi(x), \quad u(0,x) = \phi(x),$$

$(p \in (1,2])$ by changing the branching systems and scaling.

Super-Brownian motion

Now, we give an important properties of SBM.

Theorem C

1. (Konno-Shiga '88, Reimers '89) When $d = 1$ X_t is absolutely continuous with respect to Lebesgue measure for any $t > 0$ a.s. and its density $X_t(x)$ is the unique nonnegative solution to

$$\frac{\partial}{\partial t}X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}(t,x), \ \lim_{t\to 0} X_t(x)dx = X_0(dx).$$

2. (Dawson-Perkins '91, LeGall-Perkins '95) When $d \geq 2$, X_t is singular with respect to Lebesgue measure a.s. if $X_t > 0$. Also, the Hausdorff dimension of support is "2".

Branching random walks in random environment

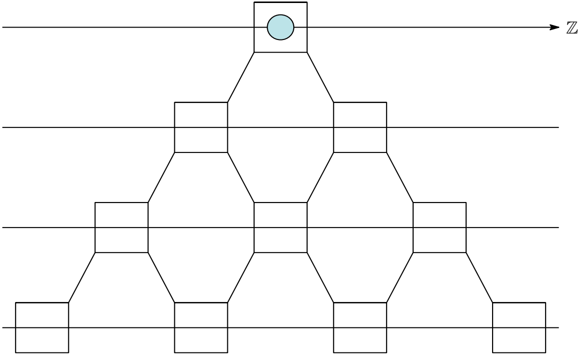
Now, we try to extend branching random walks to the model in random environment and consider the scaling limit like super-Brownian motion.

Branching random walks in random environment

1. There exist particles at $x_1, \cdots, x_{M_N} \in 2\mathbb{Z}$.
2. Each particle located at x at time n independently chooses nearest neighbor site with probability $\frac{1}{2}$ and moves there.
3. Then, it is independently replaced by two particles with probability $\frac{1}{2} + \frac{\xi(n,x)}{2N^{1/4}}$ or erased with probability with $\frac{1}{2} - \frac{\xi(n,x)}{2N^{1/4}}$, where $\{\xi(n,x))\}_{(n,x)\in\mathbb{N}\times\mathbb{Z}}$ are i.i.d. random variables taking value $\{-1,1\}$ uniformly.

Example of BRWRE

$\xi = -1$, $\xi = 1$, $N = 1$



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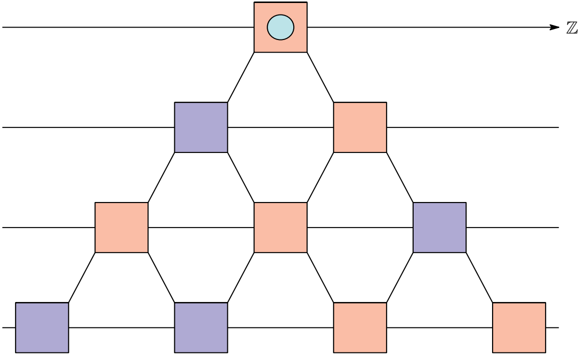

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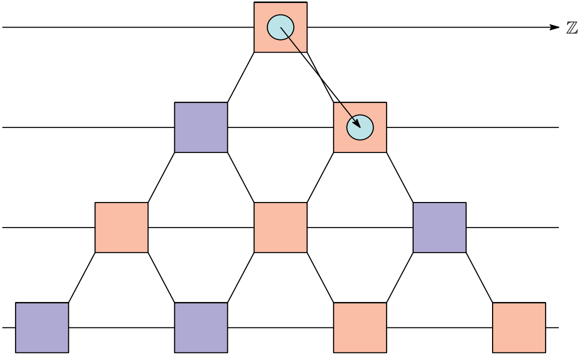

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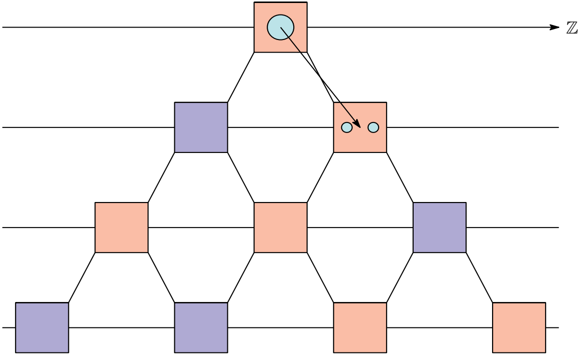

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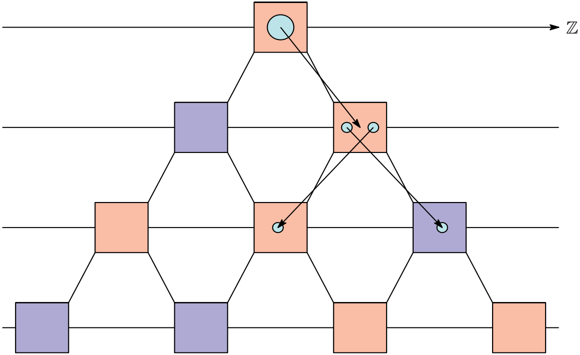

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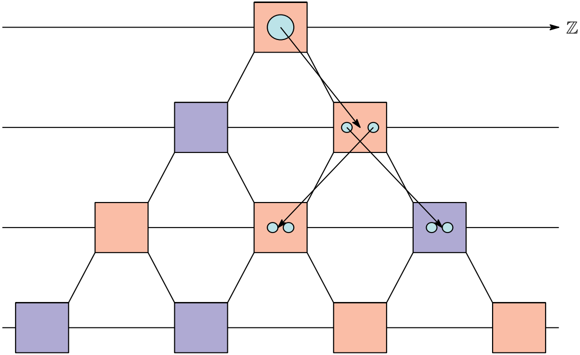

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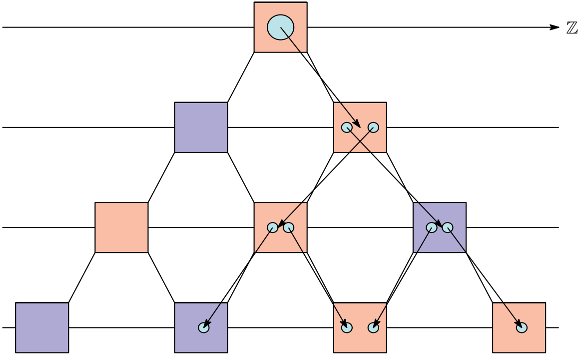

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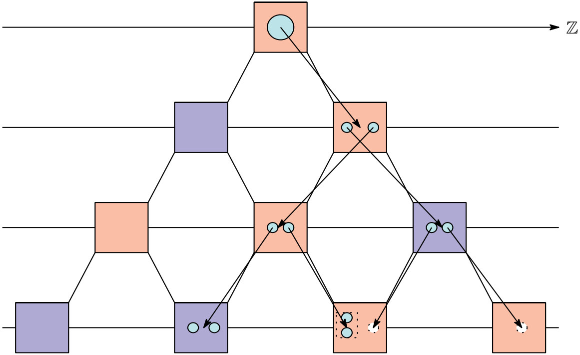

Main result

Theorem (N '14+)

Suppose $X_0^{(N)} \Rightarrow X_0$. Then, $X_\cdot^{(N)} \Rightarrow X_\cdot$. Moreover, X is absolutely continuous with respect to Lebesgue measure for any $t > 0$ a.s. and the unique nonnegative solution to the martingale problem:

$$\begin{cases} \text{For any } \phi \in C_b^2(\mathbb{R}), \\ Z_t(\phi) = X_t(\phi) - X_0(\phi) - \displaystyle\int_0^t X_s\left(\frac{1}{2}\Delta\phi\right) ds \\ \text{is a continuous and } L^2\text{-integrable } \mathcal{F}_t^X\text{-martingale such that} \\ \langle Z(\phi)\rangle_t = \displaystyle\int_0^t X_s(\phi^2) ds \\ \qquad\qquad {\color{red} +2\displaystyle\int_0^t \int_{\mathbb{R}^d} X_s^2(x)\phi^2(x) dx ds,} \end{cases}$$

where $X_t(dx) = X_t(x)dx$.

Main result

Corollary

X is the weak unique nonnegative solution to stochastic heat equation:

$$\frac{\partial}{\partial t}X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x) + 2X_t(x)^2}\dot{W}(t,x).$$
$$\lim_{t\to 0} X_t(x)dx = X_0(dx).$$

Main result

Corollary

X is the weak unique nonnegative solution to stochastic heat equation:

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2} \Delta X_t(x) + \sqrt{X_t(x) + 2X_t(x)^2} \dot{W}(t,x).$$

$$\lim_{t \to 0} X_t(x) dx = X_0(dx).$$

Thus, we have extended existence and uniqueness of nonnegative solution to finite measure initial condition for the case $\sigma(u) = \sqrt{u + 2u^2}$.

What is $N^{-1/4}$?

Intuitive reason

The summation of fluctuation of mean offsprings from 1 over $[aN^{1/2}, bN^{1/2}]$ is

$$\sum_{x\in[aN^{1/2},bN^{1/2}]} \frac{\xi(n,x)}{N^{1/4}}$$

and central limit theorem implies that it converges to normal random variable $N(0,(b-a))$.

Uniqueness

Weak uniqueness of X follows from the existence of the process Y. independent of X such that for any $\phi \in C_b^{2,+}(\mathbb{R})$

$$E_X[\exp(-X_t(\phi))] = E_Y[\exp(-X_0(Y_t))].$$

Indeed, if X and X' are nonnegative solutions, then we have that for any $\phi \in C_b^{2,+}(\mathbb{R})$,

$$E_X[\exp(-X_t(\phi))] = E_Y[\exp(-X_0(Y_t))] = E_{X'}[\exp(-X'_t(\phi))].$$

Uniqueness

Such $Y_\cdot$ is a solution to a stochastic heat equation:

$$\frac{\partial}{\partial t} Y_t(x) = \frac{1}{2}\Delta Y_t(x) - \frac{1}{2} Y_t^2(x) + \sqrt{2} Y_t(x) \dot{\tilde{W}}(t,x),$$
$$Y_0(x) = \phi(x),$$

where $\tilde{W}$ is time-space white noise independent of W (the existence of solutions follows from Dawson-Girsanov transformation).

Higher dimension?

We have a question.

Can we prove the existence of the non-trivial scaling limit of $X_t^{(N)}$ for $d \geq 2$?

Higher dimension?

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Can we prove the existence of the non-trivial scaling limit of $X_t^{(N)}$ for $d \geq 2$?

The answer is "no".

General $\sigma(u)$

Can we construct solutions to stochastic heat equation with finite measure initial condition for σ other than $\sigma = \sqrt{u + 2u^2}$?
One of idea is a change of environment for SBMRE as follows:

1. There exist particles at $x_1, \cdots, x_{M_N} \in 2\mathbb{Z}$.
2. Each particle located at x at time n independently chooses nearest neighbor site with probability $\frac{1}{2}$ and moves there.
3. Then, it is independently replaced by two particles with probability $\frac{1}{2} + \frac{\xi(n,x)}{2N^{1/4}} g\left(\frac{B_{n,x}}{N^{1/2}}\right)$ or erased with probability with $\frac{1}{2} - \frac{\xi(n,x)}{2N^{1/4}} g\left(\frac{B_{n,x}}{N^{1/2}}\right)$, where $\{\xi(n,x))\}_{(n,x)\in\mathbb{N}\times\mathbb{Z}}$ are i.i.d. random variables taking value $\{-1, 1\}$ uniformly,

where $g(u) = \frac{\tilde{\sigma}(u)}{u} 1\{u > 0\}$.

Then, "the scaling limit" will be a solution to SHE with
$\sigma(u) = \sqrt{u + |\tilde{\sigma}(u)|^2}$.

Thank you for your attentions!