Stochastic heat equation arising from a certain branching systems in random environment

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In this talk, we will consider the stochastic heat equations on the line which have been studied for four decades. Especially, we will construct a non-negative solution to a certain stochastic heat equation by using a branching systems in random environment.

1 Stochastic heat equation

In this talk, we consider the stochastic heat equations as follows:

$$\frac{\partial}{\partial t}X_t = \frac{1}{2}\Delta X_t(x) + a(X_t(x))\dot{W}(t,x), \tag{1.1}$$

where W is a time-space white noise and a is a continuous function with $a(0) = 0$.

The study of stochastic heat equation was started around 1970's. In particular, the existence and the uniqueness of the strong solution to (1.1) are known if a is Lipschitz continuous [8] et.al.

Also, the existence of the solution to (1.1) are verified for more general a under some initial conditions [7]. On the other hand, the uniqueness of solutions to (1.1) are very difficult problem attacked by many mathematicians [4, 3] et. al.

The stochastic heat equations (1.1) appear as some limit process. One of the most famous examples is a one-dimensional super-Brownian motion which is a measure-valued process arising as a scaling limit of some critical branching Brownian motion or branching random walks.

2 Super-Brownian motion

Before giving a definition of super-Brownian motion, we recall the branching random walks.*

Definition 1. *Branching random walks are defined as follows:*

(1) There are particles at $x_1, \cdots, x_{M_N} \in \mathbb{Z}^d$ at time 0.

(2) The particles at time n choose a nearest neighbor site independently and uniformly, and move there.

(3) Then, each of them independently splits into two particles with probability $\frac{1}{2}$ or vanishes with probability $\frac{1}{2}$.

Remark: The total number at time n, B_n, is a critical Galton-Watson process.

We set a measure-valued process $\{X_t^{(N)}\}$ as follows: For every Borel set A

$$X_0^{(N)}(dx) = \frac{1}{N}\sum_{i=1}^{M_N}\delta_{x_i/N^{1/2}}(dx),$$
$$X_t^{(N)}(A) = \frac{1}{N}\sharp\{\text{particles locates in } N^{1/2}A \text{ at time } \lfloor Nt \rfloor\}.$$

Then, we have the following theorem:

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*In this talk, we consider the most simple case.

Theorem 2. *([9, 1]) If $X_0^{(N)} \Rightarrow X_0$ in $\mathcal{M}_F(\mathbb{R}^d)$, then $\{X_\cdot^{(N)}\}$ weakly converges to a measure valued process X_t as $N \to \infty$.*

Moreover,[2, 6] if $d = 1$, then X_t is absolutely continuous with respect to the Lebesgue measure for any $t >$ a.s. and its density $X_t(x)$ is the unique non-negative weak solution to the stochastic heat equation

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}(t,x), \quad \lim_{t\to 0} X_t(x)dx = X_0(dx).$$

3 Main result

We construct a solution to (1.1) with $a(u) = \sqrt{u}$ from a certain branching system in random environment.

Theorem 3. *([5]) For any $X_0 \in \mathcal{M}_F(\mathbb{R})$, there exists the unique, weak, and non-negative solution to the stochastic heat equation*

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x) + X_t(x)^2}\dot{W}(t,x), \quad \lim_{t\to \infty} X_t(x)dx = X_0(dx).$$

Remark: Mytnik gave a remark on the above construction in his paper.

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