Exact convergence rate of the Wong-Zakai approximation to RDEs driven by Gaussian rough paths

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Oct. 15, 2014

How fast does an approxi. error converge?

Let

- Y: sol. to SDE driven by fBm,
- $\{Y^{\mathrm{WZ}(m)}\}_{m=1}^{\infty}$: the Wong-Zakai approxi. to Y.

Problem

Find $\{r_m\}_{m=1}^{\infty}$ s.t. $r_m \to 0$ and

$$\boldsymbol{E}[\|Y^{\mathrm{WZ}(m)} - Y\|_{\infty;[0,1]}^2]^{1/2} \leqslant r_m.$$

For a conti. func. $z : [0,1] \to \mathbf{R}^e$, we put

$$\|z\|_{\infty;[0,1]} = \sup_{t\in[0,1]} |z_t|.$$

Consider a Strat. type SDE driven by Bm

Consider an SDE

$$\begin{cases} dY_t = \sigma(Y_t)\, d^{\circ} B_t, \quad t \in (0,1], \\ \ Y_0 = y_0, \end{cases}$$

where

- B : d-dim. Bm on $(\Omega, \mathcal{F}, \boldsymbol{P})$,
- $d^{\circ}B$: the Stratonovich integral,
- $\sigma \in C^{\infty}_{\mathrm{bdd}}(\mathbf{R}^e; \mathrm{Mat}(e, d))$,
- $y_0 \in \mathbf{R}^e$

The Wong-Zakai approxi. for the Strat. SDE

- Set $\tau_k^m = k2^{-m}$ for $k = 0, \ldots, 2^m$.
- Consider an approximation $B(m)$ to B:

$$B(m)_t = (B_{\tau_k^m} - B_{\tau_{k-1}^m})2^m(t - \tau_{k-1}^m) + B_{\tau_{k-1}^m}$$

  for $\tau_{k-1}^m \leqslant t \leqslant \tau_k^m$.
- Define the Wong-Zakai approximation $Y^{\mathrm{WZ}(m)}$ by a sol. to an SODE

$$Y_t^{\mathrm{WZ}(m)} = y_0 + \int_0^t \sigma(Y_u^{\mathrm{WZ}(m)})\, dB(m)_u,$$

  where $dB(m)$ is the Riemann-Stieltjes integral.

Convergence rates

Theorem

$\forall r \geqslant 1$, $\exists C > 0$: *independent of m and*

$$\boldsymbol{E}[\|Y^{WZ(m)} - Y\|^r_{\infty;[0,1]}]^{1/r} \leqslant C 2^{-m/2}(1+m)^{1/2}.$$

FBm is a typical example of non-martingale

Definition

FBm B is a conti. centered Gaussian proc. with

$$\boldsymbol{E}[B_s B_t] = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H})$$

for some $0 < H < 1$.

Note

- if $H = 1/2$, then the fBm is a standard Bm.
- otherwise the fBm is not even a martingale.

Signature of x

Let

- $x \in C^{1\text{-var}}([0,1];\mathbf{R}^d)$,
- $\triangle = \{(s,t); 0 \leqslant s < t \leqslant 1\}$,
- $T^N(\mathbf{R}^d) = \mathbf{R} \oplus (\mathbf{R}^d) \oplus (\mathbf{R}^d)^{\otimes 2} \oplus \cdots \oplus (\mathbf{R}^d)^{\otimes N}$.

Define $\boldsymbol{x} : \triangle \to T^N(\mathbf{R}^d)$ by

$$\boldsymbol{x}_{st}^{\alpha_1\cdots\alpha_n} = \int_{s<u_1<\cdots<u_n<t} dx_{u_1}^{\alpha_1} \cdots dx_{u_n}^{\alpha_n},$$

$$\boldsymbol{x}^n = \left(\boldsymbol{x}^{\alpha_1\cdots\alpha_n}\right)_{\alpha_1\cdots\alpha_n \in \{1,\dots,d\}^n},$$

$$\boldsymbol{x} = (\boldsymbol{x}^0 \equiv 1, \boldsymbol{x}^1, \dots, \boldsymbol{x}^N).$$

We call $S_N(x) \equiv \boldsymbol{x}$ the step-N signature of x.

Finite p-variation and the Chen identity

The signature $\boldsymbol{x}$ has following properties;

- Finite p-variation:

$$\|\boldsymbol{x}\|^p_{p\text{-var};[s,t]} = \sum_{n=1}^{N} \sup_{s=\tau_0<\cdots<\tau_k=t} \sum_{l=1}^{k} |\boldsymbol{x}^n_{\tau_{l-1}\tau_l}|^{p/n}_{(\mathbf{R}^d)^{\otimes n}} < \infty$$

  for any $p \geqslant 1$.

- The Chen identity:

$$\boldsymbol{x}_{st} = \boldsymbol{x}_{st_1} *_{T^N(\mathbf{R}^d)} \boldsymbol{x}_{t_1 t}$$

  for $0 \leqslant s \leqslant t_1 \leqslant t \leqslant 1$.

The Chen identity

The Chen identity is equivalent to

$$
\begin{aligned}
\boldsymbol{x}_{st}^{\alpha} &= \boldsymbol{x}_{st_1}^{\alpha} + \boldsymbol{x}_{t_1 t}^{\alpha}, \\
\boldsymbol{x}_{st}^{\alpha_1\alpha_2} &= \boldsymbol{x}_{st_1}^{\alpha_1\alpha_2} + \boldsymbol{x}_{st_1}^{\alpha_1}\boldsymbol{x}_{t_1 t}^{\alpha_2} + \boldsymbol{x}_{t_1 t}^{\alpha_1\alpha_2}, \\
\boldsymbol{x}_{st}^{\alpha_1\alpha_2\alpha_3} &= \boldsymbol{x}_{st_1}^{\alpha_1\alpha_2\alpha_3} + \boldsymbol{x}_{st_1}^{\alpha_1\alpha_2}\boldsymbol{x}_{t_1 t}^{\alpha_2} + \boldsymbol{x}_{st_1}^{\alpha_1}\boldsymbol{x}_{t_1 t}^{\alpha_2\alpha_3} + \boldsymbol{x}_{t_1 t}^{\alpha_1\alpha_2\alpha_3}, \\
&\vdots \\
\boldsymbol{x}_{st}^{\alpha_1\cdots\alpha_N} &= \boldsymbol{x}_{st}^{\alpha_1\cdots\alpha_N} + \boldsymbol{x}_{st_1}^{\alpha_1\cdots\alpha_{N-1}}\boldsymbol{x}_{t_1 t}^{\alpha_N} \\
&\qquad + \cdots + \boldsymbol{x}_{st_1}^{\alpha_1}\boldsymbol{x}_{t_1 t}^{\alpha_2\cdots\alpha_N} + \boldsymbol{x}_{t_1 t}^{\alpha_1\cdots\alpha_N}
\end{aligned}
$$

Splitting region of the integration, we have

$$
\begin{aligned}
\boldsymbol{x}_{st}^{\alpha_1\alpha_2} &= \int_{s<u_1<u_2<t} dx_{u_1}^{\alpha_1} dx_{u_2}^{\alpha_2} \\
&= \left(\int_{s<u_1<u_2<t_1} + \int_{s<u_1<t, t_1<u_2<t} + \int_{t_1<u_1<u_2<t} \right) dx_{u_1}^{\alpha_1} dx_{u_2}^{\alpha_2} \\
&= \boldsymbol{x}_{st_1}^{\alpha_1\alpha_2} + \boldsymbol{x}_{st_1}^{\alpha_1} \boldsymbol{x}_{t_1 t}^{\alpha_2} + \boldsymbol{x}_{t_1 t}^{\alpha_1\alpha_2}.
\end{aligned}
$$

The space of geometric rough path

For $2 \leqslant p < \infty$, we define $\left(G\Omega_p(\mathbf{R}^d), \rho_{p\text{-var}}\right)$ by

$$G\Omega_p(\mathbf{R}^d) = \overline{\{S_{\lfloor p \rfloor}(x); x \in C^{1\text{-var}}([0,1];\mathbf{R}^d)\}}^{\rho_{p\text{-var}}},$$

$$\rho_{p\text{-var}}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \max_{1 \leqslant n \leqslant \lfloor p \rfloor} \rho^{(n)}_{p\text{-var}}(\boldsymbol{x}, \tilde{\boldsymbol{x}}),$$

$$\rho^{(n)}_{p\text{-var}}(\boldsymbol{x}, \tilde{\boldsymbol{x}})$$

$$= \sup_{0=\tau_0<\cdots<\tau_k=1} \left(\sum_{l=1}^{k} |\boldsymbol{x}^n_{\tau_{l-1}\tau_l} - \tilde{\boldsymbol{x}}^n_{\tau_{l-1}\tau_l}|^{p/n}_{(\mathbf{R}^d)^{\otimes n}} \right)^{n/p}.$$

Solutions to RDEs

- $\forall x \in C^{1\text{-var}}([0,1];\mathbf{R}^d)$, $\exists! y \in C([0,1];\mathbf{R}^e)$ s.t. the sol. to an ODE

$$(1) \qquad dy_t = \sigma(y_t)\,dx_t, \qquad y_0 : \text{given}.$$

- Let $\boldsymbol{x} = S_{\lfloor p \rfloor}(x)$ for some $x \in C^{1\text{-var}}([0,1];\mathbf{R}^d)$. Then the sol. $y \in C([0,1];\mathbf{R}^e)$ to an RDE

$$(2) \qquad dy_t = \sigma(y_t)\,d\boldsymbol{x}_t, \qquad y_0 : \text{given}$$

is defined by the sol. to (1).

- What is a sol. to (2) for $\boldsymbol{x} \in G\Omega_p(\mathbf{R}^d)$?

Existence of sol. and conti. of the sol. map

Theorem (Lyons, Friz-Victoir)

$\forall \boldsymbol{x} \in G\Omega_p(\mathbf{R}^d)$, $\exists \{x^{(m)}\}_{m=1}^{\infty} \subset C^{1\text{-}var}([0,1];\mathbf{R}^d)$ s.t.

$$\boldsymbol{x}^{(m)} = S_{\lfloor p \rfloor}(x^{(m)}) \xrightarrow[m\to\infty]{} \boldsymbol{x}.$$

Furthermore,

- *$\exists y \in C([0,1];\mathbf{R}^e)$ s.t.*

$$\|y^{(m)} - y\|_{\infty;[0,1]} \xrightarrow[m\to\infty]{} 0.$$

- *y is independent of $\{x^{(m)}\}$.*

The Wong-Zakai approxi. and sol. to RDE

Recall that $y^{\mathrm{WZ}(m)}$ is a sol. to an ODE

$$dy_t^{\mathrm{WZ}(m)} = \sigma(y_t^{\mathrm{WZ}(m)})\,dx(m)_t,$$

where $x(m)$ isthe dyadic polygonal approxi. to x.
Assume that $\boldsymbol{x} \in G\Omega_p(\mathbf{R}^d)$ satisfy

$$\boldsymbol{x}(m) = S_{\lfloor p \rfloor}(x(m)) \xrightarrow[m\to\infty]{} \boldsymbol{x}.$$

From the theorem, we have

$$\|y^{\mathrm{WZ}(m)} - y\|_{\infty;[0,1]} \xrightarrow[m\to\infty]{} 0.$$

Consider an SDE driven by fBm

Let

- B : d-dim. fBm with $1/3 < H < 1/2$,
- $\boldsymbol{B}(m)$: the natural GRP associated to $B(m)$.

1. If $p > 1/H$, then $\exists \boldsymbol{B} \in G\Omega_p(\mathbf{R}^d)$ s.t.

$$\lim_{m\to\infty} \rho_{p\text{-var}}(\boldsymbol{B}(m), \boldsymbol{B}) = 0$$

in L^r and a.s.

2. The SDE

$$\begin{cases} dY_t = \sigma(Y_t)\, d\boldsymbol{B}_t, & t \in (0, 1], \\ Y_0 = y_0, \end{cases}$$

has a sol. in the RDE sense.

The conv. rate of WZ approxi. is $2^{-m(2H-1/2)}$

Theorem (N)

Let $1/3 < H < 1/2$*. Then*

- $\forall r \geqslant 1$, $\exists C > 0$*: independent of m and*

$$\boldsymbol{E}[\|Y^{WZ(m)} - Y\|^r_{\infty;[0,1]}]^{1/r} \leqslant C2^{-m(2H-1/2)}.$$

- *The above rate is optimal.*

Remark

The first assertion is valid for Gaussian drivers satisfying the Coutin-Qian cond. for $1/3 < \lambda < 1/2$

The Coutin-Qian condition

Let $X = (X^1, \dots, X^d)$ be a conti. centered Gaussian proc. with IID components.

Definition

We say that X satisfies the Coutin-Qian cond. for $0 < \lambda < 1$ if $\exists C > 0$ s.t.

- $\boldsymbol{E}[(X_t^\alpha - X_s^\alpha)^2] \leqslant C(t-s)^{2\lambda}$ for $0 \leqslant s < t \leqslant 1$
- $|\boldsymbol{E}[(X_{t+\epsilon}^\alpha - X_t^\alpha)(X_{s+\epsilon}^\alpha - X_s^\alpha)]| \leqslant C(t-s)^{2\lambda-2}\epsilon^2$ for $0 \leqslant s < t \leqslant 1$ and $\epsilon > 0$ with $\epsilon < t-s$.

Preceding result

Theorem (Bayer et al. 2013)

Let $1/4 < H < 1/2$ *and* $0 < \kappa < 2H - 1/2$.

Then, $\forall r \geqslant 1$, $\exists C > 0$*: independent of m and*

$$\boldsymbol{E}[\| Y^{WZ(m)} - Y\|^r_{\infty;[0,1]}]^{1/r} \leqslant C2^{-m\kappa}.$$

Remark

- The case $\kappa = 2H - 1/2$ is our main result.
- Many approxi. satisfy the above estimate.

Proof of the optimality

Consider the SDE

$$\begin{cases} dY_t^1 = dB_t^1, \\ dY_t^2 = Y_t^1 \, dB_t^2, \end{cases} \qquad Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then the sol. and the Wong-Zakai approxi. are given by

$$Y_t = \begin{pmatrix} B_t^1 \\ \boldsymbol{B}_{0,t}^{12} \end{pmatrix}, \qquad Y_t^{\mathrm{WZ}(m)} = \begin{pmatrix} B(m)_t^1 \\ \boldsymbol{B}(m)_{0,t}^{12} \end{pmatrix},$$

respectively.

In this case, we have

$$\begin{aligned}\boldsymbol{E}[\|Y^{\mathrm{WZ}(m)} - Y\|_{\infty;[0,1]}^2] &\geqslant \boldsymbol{E}[|Y_1^{\mathrm{WZ}(m)} - Y_1|^2] \\ &\geqslant \boldsymbol{E}[|B(m)_1^1 - B_1^1|^2] + \boldsymbol{E}[|\boldsymbol{B}(m)_{0,1}^{12} - \boldsymbol{B}_{0,1}^{12}|^2].\end{aligned}$$

It is enough to prove

$$\boldsymbol{E}[|\boldsymbol{B}(m)_{0,1}^{12} - \boldsymbol{B}_{0,1}^{12}|^2] \geqslant C2^{m(4H-1)}.$$

Exact convergence rate of iterated integral

Theorem

Let $\alpha \neq \beta$. If $1/4 < H < 1/2$, then $\exists C > 0$ s.t.

$$\lim_{m \to \infty} 2^{m(4H-1)} \boldsymbol{E}[|\boldsymbol{B}(m)_{st}^{\alpha\beta} - \boldsymbol{B}_{st}^{\alpha\beta}|^2] = C(t-s)$$

for $0 \leqslant s < t \leqslant 1$.

For simplicity, we prove it for dyadic rationals $s < t$.

From the Chen identity,

$$\boldsymbol{B}_{st}^{\alpha\beta} = \sum_{k=2^m s+1}^{2^m t} \boldsymbol{B}_{\tau_{k-1}^m \tau_k^m}^{\alpha\beta} + \sum_{k=2^m s+1}^{2^m t} \boldsymbol{B}_{s\tau_{k-1}^m}^{\alpha} \boldsymbol{B}_{\tau_{k-1}^m \tau_k^m}^{\beta}$$

$$= \sum_{k=2^m s+1}^{2^m t} \boldsymbol{B}_{\tau_{k-1}^m \tau_k^m}^{\alpha\beta} + \sum_{k=2^m s+1}^{2^m t} \left(B_{\tau_{k-1}^m}^{\alpha} - B_s^{\alpha}\right)\left(B_{\tau_k^m}^{\beta} - B_{\tau_{k-1}^m}^{\beta}\right).$$

The same equality holds for $\boldsymbol{B}(m)_{st}^{\alpha\beta}$. Hence,

$$\boldsymbol{B}(m)_{st}^{\alpha\beta} - \boldsymbol{B}_{st}^{\alpha\beta} = \sum_{k=2^m s+1}^{2^m t} \left\{ \boldsymbol{B}(m)_{\tau_{k-1}^m \tau_k^m}^{\alpha\beta} - \boldsymbol{B}_{\tau_{k-1}^m \tau_k^m}^{\alpha\beta} \right\}.$$

Set

$$I_{k,l}^{(m)} = 2^{4mH}\boldsymbol{E}\left[\left\{\boldsymbol{B}(m)^{\alpha\beta}_{\tau^m_{k-1}\tau^m_k} - \boldsymbol{B}^{\alpha\beta}_{\tau^m_{k-1}\tau^m_k}\right\} \times \left\{\boldsymbol{B}(m)^{\alpha\beta}_{\tau^m_{l-1}\tau^m_l} - \boldsymbol{B}^{\alpha\beta}_{\tau^m_{l-1}\tau^m_l}\right\}\right]$$

for $0 \leqslant k \leqslant l \leqslant 2^m$.

Then, we see

1. $I_{k,l}^{(m)}$ is depending only on $l-k$, i.e. $I_{k,l}^{(m)} = I_{l-k}$,
2. $|I_{k,l}^{(m)}| \leqslant C(l-k)^{2H-2}$ for $l-k \geqslant 1$.

From the two properties above, we have

$$
\begin{aligned}
&2^{m(4H-1)}\boldsymbol{E}[|\boldsymbol{B}(m)^{\alpha\beta}_{st} - \boldsymbol{B}^{\alpha\beta}_{st}|^2] \\
&= \frac{1}{2^m}\boldsymbol{E}\left[\left\{\sum_{k=2^m s+1}^{2^m t} 2^{2mH}\left(\boldsymbol{B}(m)^{\alpha\beta}_{\tau^m_{k-1}\tau^m_k} - \boldsymbol{B}^{\alpha\beta}_{\tau^m_{k-1}\tau^m_k}\right)\right\}^2\right] \\
&= \frac{1}{2^m}\sum_{k=2^m s+1}^{2^m t} I_0 + \frac{2}{2^m}\sum_{2^m s+1\leqslant k<l\leqslant 2^m t} I_{l-k} \\
&= (t-s)\left\{I_0 + 2\sum_{j=1}^{2^m t-2^m s+1} I_j\right\} + o(1) \\
&\to (t-s)C
\end{aligned}
$$

$I_{k,l}^{(m)}$ is depending only on $l-k$

From

- the self-similarity,
- the stationary increments

of fBm, we see that $I_{k,l}^{(m)}$ is equal to

$$
\begin{aligned}
& I_{k,l}^{(m)} \\
&= \boldsymbol{E}\left[\left\{\boldsymbol{B}(m)_{k-1,k}^{\alpha\beta} - \boldsymbol{B}_{k-1,k}^{\alpha\beta}\right\}\left\{\boldsymbol{B}(m)_{l-1,l}^{\alpha\beta} - \boldsymbol{B}_{l-1,l}^{\alpha\beta}\right\}\right] \\
&= \boldsymbol{E}\left[\left\{\boldsymbol{B}(m)_{0,1}^{\alpha\beta} - \boldsymbol{B}_{0,1}^{\alpha\beta}\right\}\left\{\boldsymbol{B}(m)_{0,l-k+1}^{\alpha\beta} - \boldsymbol{B}_{0,l-k+1}^{\alpha\beta}\right\}\right] \\
&= I_{l-k}
\end{aligned}
$$

$|I_{k,l}^{(m)}| \leqslant C(l-k)^{2H-2}$ for $l-k \geqslant 1$

Proposition

Let $m', m'' \geqslant m$. Then, for any $l-k \geqslant 1$,

$$\left| \boldsymbol{E}\left[\boldsymbol{B}(m')^{\alpha\beta}_{\tau^m_{k-1}\tau^m_k} \boldsymbol{B}(m'')^{\alpha\beta}_{\tau^m_{l-1}\tau^m_l} \right] \right| \leqslant C|\tau^m_k - \tau^m_l|^{2H-2} 2^{-2mH} 2^{-2m}.$$

Hence

$$|I_{k,l}^{(m)}| \leqslant 2^{4mH} \cdot 4 \cdot C|\tau^m_k - \tau^m_l|^{2H-2} 2^{-2mH} 2^{-2m} = 4C|k-l|^{2H-2}.$$

We prove this proposition for $m = m' = m''$.

Recall

$$\boldsymbol{B}(m)^{\alpha\beta}_{\tau^m_{k-1}\tau^m_k} = \int_{\tau^m_{k-1}}^{\tau^m_k} \{B(m)^\alpha_u - B(m)^\alpha_{\tau^m_{k-1}}\} \frac{dB(m)^\beta_u}{du}\, du.$$

It follows from the Coutin-Qian condition that

$$|\boldsymbol{E}[\{B(m)^\alpha_u - B(m)^\alpha_{\tau^m_{k-1}}\} \\ \times \{B(m)^\alpha_v - B(m)^\alpha_{\tau^m_{l-1}}\}]| \leqslant C2^{-2mH}$$

and

$$\left| \boldsymbol{E}\left[\frac{dB(m)^\beta_u}{du} \frac{dB(m)^\beta_v}{dv} \right] \right| \leqslant C|\tau^m_k - \tau^m_l|^{2H-2}.$$

for $\tau^m_{k-1} \leqslant u \leqslant \tau^m_k$ and $\tau^m_{l-1} \leqslant v \leqslant \tau^m_l$.

Since B^α and B^β are indep., we have

$$\begin{aligned}
&\left|\boldsymbol{E}\left[\boldsymbol{B}(m)^{\alpha\beta}_{\tau^m_{k-1}\tau^m_k}\boldsymbol{B}(m)^{\alpha\beta}_{\tau^m_{l-1}\tau^m_l}\right]\right| \\
&\leqslant \int_{\tau^m_{k-1}}^{\tau^m_k} du \int_{\tau^m_{l-1}}^{\tau^m_l} dv \\
&\qquad \times |\boldsymbol{E}[\{B(m)^\alpha_u - B(m)^\alpha_{\tau^m_{k-1}}\}\{B(m)^\alpha_v - B(m)^\alpha_{\tau^m_{l-1}}\}]| \\
&\qquad \times \left|\boldsymbol{E}\left[\frac{dB(m)^\beta_u}{du}\frac{dB(m)^\beta_v}{dv}\right]\right| \\
&\leqslant C|\tau^m_k - \tau^m_l|^{2H-2}2^{-2mH}2^{-2m}.
\end{aligned}$$

Proof of the upper bound

The conv. rate of WZ approxi. is $2^{-m(2H-1/2)}$

Theorem (N)

Let $1/3 < H < 1/2$. *Then*

- $\forall r \geqslant 1$, $\exists C > 0$: *independent of m and*

$$\boldsymbol{E}[\|Y^{WZ(m)} - Y\|_{\infty;[0,1]}^r]^{1/r} \leqslant C2^{-m(2H-1/2)}.$$

- *The above rate is optimal.*

Keys in proof of the exact conv. rate

Theorem (N)

Let $1/3 < H < 1/2$ *and* $p > \frac{1}{1/2-H}$.
Then $\forall r \geqslant 1$, $\exists C > 0$*: independent of* m *and*

$$\boldsymbol{E}[\rho_{p\text{-}var}(\boldsymbol{B}(m), \boldsymbol{B})^r]^{1/r} \leqslant C2^{-m(2H-1/2)}.$$

Theorem (Bayer et al. 2013)

$$\|Y^{WZ(m)} - Y\|_{\infty;[0,1]} \leqslant G\rho_{p\text{-}var}(\boldsymbol{B}(m), \boldsymbol{B}),$$

where G *is a r.v. satisfying* $\boldsymbol{E}[G^r] < \infty$ *for* $\forall r \geqslant 1$.

Proof of the theorem

We will prove

Theorem (N)

Let $1/3 < H < 1/2$ *and* $p > \frac{1}{1/2-H}$.
Then $\forall r \geqslant 1$, $\exists C > 0$*: independent of* m *and*

$$\boldsymbol{E}[\rho_{p\text{-}var}(\boldsymbol{B}(m), \boldsymbol{B})^r]^{1/r} \leqslant C2^{-m(2H-1/2)}.$$

Step 1

We prove the following key estimate:

Proposition

Let $n = 1, 2, 3$. Then $\exists C_n > 0$ s.t.

$$\boldsymbol{E}[|\boldsymbol{B}(m)_{st}^{\alpha_1\cdots\alpha_n} - \boldsymbol{B}_{st}^{\alpha_1\cdots\alpha_n}|^2]^{1/2} \leqslant C_n 2^{-m(2H-1/2)}(t-s)^{1/2-H}(t-s)^{(n-1)H}$$

for $s = \tau_k^m$ and $t = \tau_l^m$ with $k < l$.

Step 2

From the Hölder conti. of $\boldsymbol{B}$, we have

Proposition

Let $n = 1, 2, 3$. *Then* $\exists C_n > 0$ *s.t.*

$$\begin{aligned} \boldsymbol{E}[|\boldsymbol{B}(m)_{st}^{\alpha_1 \cdots \alpha_n} - \boldsymbol{B}_{st}^{\alpha_1 \cdots \alpha_n}|^2]^{1/2} \\ \leqslant C_n (2^{-m} \wedge (t-s))^{2H-1/2} \\ \times (t-s)^{1/2-H} (t-s)^{(n-1)H} \end{aligned}$$

for $0 \leqslant s < t \leqslant 1$

Step 3

From the Lyons extension theorem (Friz-Riedel 2014), we have

Proposition

$\forall n, \exists C_n > 0$ *s.t.*

$$\boldsymbol{E}[|\boldsymbol{B}(m)_{st}^{\alpha_1\cdots\alpha_n} - \boldsymbol{B}_{st}^{\alpha_1\cdots\alpha_n}|^2]^{1/2} \leqslant C_n(2^{-m} \wedge (t-s))^{2H-1/2} \times (t-s)^{1/2-H}(t-s)^{(n-1)H}$$

for $0 \leqslant s < t \leqslant 1$

Step 4

Noting $1/2 - H < H$, we see

$$\boldsymbol{E}[|\boldsymbol{B}(m)_{st}^{\alpha_1\cdots\alpha_n} - \boldsymbol{B}_{st}^{\alpha_1\cdots\alpha_n}|^2]^{1/2} \leqslant C_n 2^{-m(2H-1/2)}(t-s)^{n(1/2-H)}.$$

By using the Kolmogorov criterion, we obtain

$$\boldsymbol{E}[\rho_{p\text{-var}}(\boldsymbol{B}(m), \boldsymbol{B})^r]^{1/r} \leqslant C2^{-m(2H-1/2)}$$

for any $r \geqslant 1$ and $p > \frac{1}{1/2-H}$.

Thank you for your attention.