Exact convergence rate of the Wong-Zakai approximation to RDEs driven by Gaussian rough paths

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Abstract

We consider a solution to a stochastic differential equation (SDE) driven by a Gaussian process in the sense of rough differential equation (RDE) and the Wong-Zakai approximation to the solution. We give an upper bound of the error of the Wong-Zakai approximation. We also show that the upper bound is optimal in a particular case.

1 Introduction

The rough path theory originated from Lyons gives a framework that allows us to deal with differential equations driven by rough signals rigorously. After Lyons' work, many researchers apply it to SDEs. In context of SDEs, the rough path theory plays a crucial role in order to study differential equations with rougher driving signals than Brownian motion; for example, fractional Brownian motions and more general Gaussian processes.

A key step to consider SDEs in the rough path theory is to construct rough paths associated to the drivers. Coutin-Qian showed an existence of a rough path associated to a Gaussian process under the condition so-called the Coutin-Qian condition (Defintion 1) and Friz-Victoir proved an existence under more mild conditions on its covariance function. Once we construct the rough paths associated to the drivers of SDEs, we obtain solutions to SDEs automatically with help of the rough path theory. Moreover, we can obtain a pathwise estimate for the difference of two solutions to SDEs which have different drivers; by using the local Lipschitz continuity of the Itô-Lyons map, we see that the difference of the solutions inherit from the difference of the drivers. However, we need to make an effort to obtain a probabilistic error bounds. Since the Lipschitz constant appeared in the Itô-Lyons map is a random variable, we need to consider integrability of it. The integrability is proved by [CLL13, FR13, BFRS13].

In this talk, combing the integrability of the Lipschitz constant stated above and the estimates of two rough paths (Theorem 2), we obtain the exact convergence rates of the approximations to SDE (Theorem 3).

2 Main results

Let $X = (X^1, \dots, X^d)$ be a continuous, centered d-dimensional Gaussian process with independent and identically distributed components. We assume that X satisfies the following the Coutin-Qian condition:

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Definition 1. We say that X satisfies the Coutin-Qian conditions for $0 < \lambda < 1$ if there exists a positive constant $\mathcal{C}_\lambda$ such that

$$\boldsymbol{E}\left[(X_t^\alpha - X_s^\alpha)^2\right] \le \mathcal{C}_\lambda |t-s|^{2\lambda} \text{ for any } 0 < s, t < 1,$$
$$\left|\boldsymbol{E}\left[(X_{s+\epsilon}^\alpha - X_s^\alpha)(X_{t+\epsilon}^\alpha - X_t^\alpha)\right]\right| \le \mathcal{C}_\lambda |t-s|^{2\lambda-2}\epsilon^2 \text{ for any } 0 < \epsilon < |t-s|.$$

We define the m-th dyadic polygonal approximation $X(m)$ to X by

$$X(m)_t = (X_{\tau_k^m} - X_{\tau_{k-1}^m})2^m(t - \tau_{k-1}^m) + X_{\tau_{k-1}^m}$$

for $\tau_{k-1}^m \le t \le \tau_k^m$, where $\tau_k^m = k2^{-m}$. Denote by $\boldsymbol{X}(m)$ the natural rough path associated to $X(m)$. It is known that there exists a limit rough path $\boldsymbol{X}$ in $(G\Omega_p(\mathbf{R}^d), \rho_{p\text{-var}})$ under the Coutin-Qian condition for $1/4 < \lambda \le 1/2$ and $\lambda p > 1$. Here $G\Omega_p(\mathbf{R}^d)$ is the space of geometric rough paths and $\rho_{p\text{-var}}$ is the (inhomogeneous) metric which is defined by

$$\rho_{p\text{-var}}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \max_{1\le \ell \le \lfloor p \rfloor} \rho_{p\text{-var}}^{(\ell)}(\boldsymbol{x}, \tilde{\boldsymbol{x}}),$$
$$\rho_{p\text{-var}}^{(\ell)}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_k=1} \left(\sum_{l=1}^k |\boldsymbol{x}_{\tau_{l-1}\tau_l}^\ell - \tilde{\boldsymbol{x}}_{\tau_{l-1}\tau_l}^\ell|_{(\mathbf{R}^d)^{\otimes \ell}}^{p/\ell} \right)^{\ell/p}.$$

Under this setting, we obtain the following:

Theorem 2. *Assume that X satisfies the Coutin-Qian condition for $1/3 < \lambda < 1/2$. Then*

$$\boldsymbol{E}[|\rho_{p\text{-}var}(\boldsymbol{X}, \boldsymbol{X}(m))|^r]^{1/r} \le C 2^{-m(2\lambda - 1/2)}$$

for any $r \ge 1$ and $p > 1/(1/2 - \lambda)$.

Theorem 3. *Let $\sigma \in C_{bdd}^\infty(\mathbf{R}^e; \mathbf{R}^d \otimes \mathbf{R}^e)$. Assume that X satisfies the Coutin-Qian condition for $1/3 < \lambda < 1/2$. Consider the solutions to SDEs*

$$\begin{cases} dY_t = \sigma(Y_t)\, dX_t, \\ Y_0 = y_0 \in \mathbf{R}^e, \end{cases} \qquad \begin{cases} dY(m)_t = \sigma(Y(m)_t)\, dX(m)_t, \\ Y(m)_0 = Y_0 \in \mathbf{R}^e. \end{cases}$$

Then, for any $r \ge 1$, there exists a positive constant C independent of m such that satisfy

$$\boldsymbol{E}\left[\left(\sup_{0 \le t \le 1} |Y_t - Y(m)_t| \right)^r \right]^{1/r} \le C 2^{-m(2\lambda - 1/2)}.$$

References

[BFRS13] C. Bayer, P. Friz, S. Riedel, and J. Schoenmakers. From rough path estimates to multilevel Monte Carlo. 2013.

[CLL13] T. Cass, C. Litterer, and T. Lyons. Integrability and tail estimates for Gaussian rough differential equations. *Ann. Probab.*, 41(4):3026–3050, 2013.

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