Selfadjoint extensions of a Schrödinger-type operator

Jun Masamune
Tohoku University, GSIS

Let $M = (M, g)$ be a smooth Riemannian manifold without boundary, not necessarily geodesically complete. We consider a Schrödinger-type operator

$$L = \Delta + V$$

where $\Delta = \mathrm{div} \circ \nabla$ is the Laplace-Beltrami operator and V is a real-valued continuous function on M. By Green's formula, L is a symmetric operator on the space of smooth functions with compact support $C_0^\infty(M)$ in $L^2 = L^2(M; dv_g)$, that is,

$$(Lu, v) = (u, Lv), \qquad \forall u, v \in C_0^\infty(M)$$

where (u, v) stands for the L^2-inner product of u and v. In this talk we will discuss several problems regarding with selfadjoint extensions of L; in particular, the essential selfadjointness, i.e., L has unique selfadjoint extension, as well as the Markov uniqueness by which we mean that there exists unique selfadjoint extension $\hat{L}$ satisfying the Markov property:

$$0 \le u \le 1 \quad \Rightarrow \quad 0 \le T_t u \le 1, \quad \forall t > 0 \tag{1}$$

where $\{T_t\}_{t\ge 0}$ is the L^2-semigroup generated by $\hat{L}$. Roughly speaking, (1) means that the system should not increase the energy with time $t > 0$, and together with an additional assumption[1], $\{T_t\}_{t\ge 0}$ is a transition probability of a Markov process (actually, a Hunt process) on M due to Fukushima's theorem.

Therefore, these two problems are to determine the possible physical models in quantum mechanics and Markov processes respectively, or more precisely, to study the negligibility of the singular sets of the space by these systems.

Our starting point is M.P. Gaffney [3] proving that Δ has unique Markov extension provided M is geodesically complete. This result has been extended to various directions: P.R. Chernoff [4] and R.S. Strichartz [6] showed the essential self-adjointness of Δ for geodesically complete manifolds. See [2, 5, 8, 7, 9] for results on geodesically incomplete manifolds. Another direction is to study the behavior of V and the geometry of M so that L is essentially selfadjoint (see, e.g., [1] and the reference within).

[1]The domain $\mathcal{F} = \mathrm{Dom}(\sqrt{-L})$ of the energy $\mathcal{E}$ satisfies that $\mathcal{F} \cap C_\infty(M)$ is dense in $\mathcal{F}$ and $C_\infty(M)$ with respect to $\|u\| = \sqrt{\mathcal{E}(u,u) + (u,u)}$ and sup-norm, respectively.

The main discussion of the talk will be devoted to some new results along those lines for geodesically incomplete manifolds when the singular points enjoy a certain symmetry. Time permitting, we will also address a recent result for

$$L = \Delta + X + V$$

where $X \in \Gamma(TM)$ answering the same question but it's non-symmetric counter part as a generalization of [10].

References

[1] M. Braverman, O. Milatovich, M. Shubin, Essential selfadjointness of Schrödinger-type operators on manifolds. (Russian) Uspekhi Mat. Nauk 57 (2002), no. 4(346), 3-58; translation in Russian Math. Surveys 57 (2002), no. 4, 641-692

[2] Y. Colin de Verdière, Pseudo-Laplaciens. 1, Ann. Inst. Fourier. 32 (1982), 275-286.

[3] M.P. Gaffney, A special Stokes' theorem for a complete Riemannian manifold. Ann. Math. 60 (1954), 140-145.

[4] P.R. Chernoff, Essential self-adjointness of powers of generators of hyperbolic equations. J. Funct. Anal. 12 (1973), 401-414.

[5] S. T. Kuroda, An intoduction to scattering theory, Lecture notes series No. 51, Aarhus Univ (1978).

[6] R.S. Strichartz, Analysis of the Laplacian of a complete manifold. J. Funct. Anal. 52 (1983), 48-79.

[7] J. Masamune, Essential self-adjointness of Laplacians on Riemannian manifolds with fractal boundary. Comm. Partial Differential Equations 24 (1999), no. 3-4, 749-757.

[8] P. Li and G. Tian, On the heat kernel of the Bergmann metric on algebraic varieties, J. Amer. Math. Soc. 8 (1995), 857-877.

[9] M. Nagase, On the heat operators of normal singular algebraic surfaces, J. Diff. Geom. 28 (1988), 37-57.

[10] I. Shigekawa, Non-symmetric diffusions on a Riemannian manifold. Probabilistic approach to geometry, 437-461, Adv. Stud. Pure Math., 57, Math. Soc. Japan, Tokyo, 2010.