# DIFFERENTIAL EQUATIONS DRIVEN BY ROUGH PATHS: AN APPROACH VIA FRACTIONAL CALCULUS

YU ITO

In this talk, I will consider differential equations driven by $\beta$-Hölder rough paths with $\beta \in (1/3, 1/2]$. First, on the basis of fractional calculus, I will introduce an integral of controlled paths along the rough paths (Eq. (2)). This can be regarded as an alternative approach to the integration introduced by M. Gubinelli [1]. Then, combining Eqs. (1) and (2), the solution of the differential equations will be defined by the same way introduced in [1]. Finally, as the main results of this talk, I will report the existence, uniqueness and continuity of the solution of the differential equations driven by geometric $\beta$-Hölder rough paths.

In the following, I will introduce some basic concepts which will be used in this talk.

**Notation.** Let $V$ and $W$ be finite-dimensional normed spaces. We use $L(V, W)$ to denote the set of all linear maps from $V$ to $W$. Let $T$ denote a positive constant and $\Delta_T$ denote the simplex $\{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. We use $\mathcal{C}_1^\lambda(V)$ to denote the space of all $V$-valued $\lambda$-Hölder continuous functions on the interval $[0, T]$. We use $\mathcal{C}_2(V)$ to denote the space of all $V$-valued continuous functions on $\Delta_T$. Furthermore, for $\Psi \in \mathcal{C}_2(V)$ and $\mu > 0$, we set

$$\|\Psi\|_\mu := \sup_{0 \leq s < t \leq T} \frac{\|\Psi_{s,t}\|_V}{(t-s)^\mu} \quad \text{and} \quad \mathcal{C}_2^\mu(V) := \{\Psi \in \mathcal{C}_2(V) : \|\Psi\|_\mu < \infty\}.$$

**Rough paths.** Let $\beta \in (1/3, 1/2]$. We say that a continuous map $X = (X^1, X^2)$ from $\Delta_T$ to $\mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$ is a $\beta$-Hölder rough path in $\mathbb{R}^d$ if $X$ satisfies the following properties:

(1) for each $s, t, u \in [0, T]$ with $s \leq u \leq t$,

$$X^1_{s,t} = X^1_{s,u} + X^1_{u,t} \quad \text{and} \quad X^2_{s,t} = X^2_{s,u} + X^1_{s,u} \otimes X^1_{u,t} + X^2_{u,t};$$

(2) $X^1 \in \mathcal{C}_2^\beta(\mathbb{R}^d)$ and $X^2 \in \mathcal{C}_2^{2\beta}(\mathbb{R}^d \otimes \mathbb{R}^d)$.

The space of all $\beta$-Hölder rough paths in $\mathbb{R}^d$ is denoted by $\Omega_\beta(\mathbb{R}^d)$, which is a complete metric space whose distance is

$$d_\beta(X, \tilde{X}) := \|X^1 - \tilde{X}^1\|_\beta + \|X^2 - \tilde{X}^2\|_{2\beta}$$

for $X = (X^1, X^2), \tilde{X} = (\tilde{X}^1, \tilde{X}^2) \in \Omega_\beta(\mathbb{R}^d)$. Let $x \in \mathcal{C}_1^1(\mathbb{R}^d)$. We set

$$X^1_{s,t} := x_t - x_s \quad \text{and} \quad X^2_{s,t} := \int_s^t (x_u - x_s) \otimes dx_u$$

for $(s,t) \in \Delta_T$. Then we see that $X = (X^1, X^2)$ is a $\beta$-Hölder rough paths in $\mathbb{R}^d$. This is called smooth rough path or the step-2 signature of $x$. The elements in the closure of the set of all smooth rough paths with respect to the distance $d_\beta$ are called geometric $\beta$-Hölder rough paths. The spaces of all smooth rough paths and geometric $\beta$-Hölder rough paths in $\mathbb{R}^d$ are denoted by $S\Omega_\beta(\mathbb{R}^d)$ and $G\Omega_\beta(\mathbb{R}^d)$, respectively.

**Controlled paths.** Let $X = (X^1, X^2) \in \Omega_\beta(\mathbb{R}^d)$. We say that a pair $(Y, Y')$ is an $\mathbb{R}^e$-valued controlled path based on $X$ if $(Y, Y')$ satisfies the following properties:

(1) $Y \in \mathcal{C}_1^\beta(\mathbb{R}^e)$ and $Y' \in \mathcal{C}_1^\beta(L(\mathbb{R}^d, \mathbb{R}^e))$;

(2) $R^Y \in \mathcal{C}_2^{2\beta}(\mathbb{R}^e)$, where $R^Y_{s,t} := Y_t - Y_s - Y'_s X^1_{s,t}$ for $(s,t) \in \Delta_T$.

The space of all $\mathbb{R}^e$-valued controlled paths based on $X$ is denoted by $\mathcal{Q}_X^\beta(\mathbb{R}^e)$, which is a Banach space whose norm is

$$\|(Y, Y')\|_{X,\beta} := |Y_0| + |Y'_0| + \|R^Y\|_{2\beta} + \|\delta Y'\|_\beta, \quad (Y, Y') \in \mathcal{Q}_X^\beta(\mathbb{R}^e).$$

Here $\delta Y'_{s,t} := Y'_t - Y'_s$ for $(s,t) \in \Delta_T$. Let $f$ be an $L(\mathbb{R}^d, \mathbb{R}^e)$-valued continuously Fréchet differentiable function on $\mathbb{R}^e$ whose derivative $\nabla f$ is Lipschitz continuous on $\mathbb{R}^e$. We set

$$Z_t := f(Y_t) \quad \text{and} \quad Z'_t := \nabla f(Y_t) Y'_t \tag{1}$$

for $t \in [0, T]$. Then we see that $(Z, Z')$ belongs to $\mathcal{Q}_X^\beta(L(\mathbb{R}^d, \mathbb{R}^e))$.

**Integration of controlled paths via fractional calculus.** Let $\Psi \in \mathcal{C}_2^\lambda(V)$ with $0 < \lambda \leq 1$. For $\alpha \in (0, \lambda)$, $s \in [0, T)$ and $t \in (0, T]$, we define $\mathcal{D}^\alpha_{s+}\Psi$ and $\mathcal{D}^\alpha_{t-}\Psi$ as $\mathcal{D}^\alpha_{s+}\Psi(s) := 0$,

$$\mathcal{D}^\alpha_{s+}\Psi(u) := \frac{1}{\Gamma(1-\alpha)} \left( \frac{\Psi_{s,u}}{(u-s)^\alpha} + \alpha \int_s^u \frac{\Psi_{v,u}}{(u-v)^{\alpha+1}} \, dv \right) \quad \text{for } u \in (s, T]$$

and $\mathcal{D}^\alpha_{t-}\Psi(t) := 0$,

$$\mathcal{D}^\alpha_{t-}\Psi(r) := \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)} \left( \frac{\Psi_{r,t}}{(t-r)^\alpha} + \alpha \int_r^t \frac{\Psi_{r,v}}{(v-r)^{\alpha+1}} \, dv \right) \quad \text{for } r \in [0, t),$$

where $\Gamma$ is the Euler gamma function. If $\Psi$ is of the form $\Psi_{s,t} = \psi_t - \psi_s$ for some $\psi \in \mathcal{C}_1^\lambda(V)$, then, from the definition, these functions coincide with the Weyl–Marchaud fractional derivatives of $\psi$ of order $\alpha$. Moreover, for $X = (X^1, X^2) \in \Omega_\beta(\mathbb{R}^d)$, $(Z, Z') \in \mathcal{Q}_X^\beta(L(\mathbb{R}^d, \mathbb{R}^e))$ and $\gamma \in ((1-\beta)/2, \beta)$, an $\mathbb{R}^e$-valued function $I^\gamma(X, Z)$ on $\Delta_T$ is defined by

$$\begin{aligned} I^\gamma(X, Z)_{s,t} := Z_s X^1_{s,t} + Z'_s X^2_{s,t} &+ (-1)^{1-\gamma} \int_s^t \mathcal{D}^{1-\gamma}_{s+} R^Z(u) \mathcal{R}^{(1,\gamma)}_{t-} X(u) \, du \\ &+ (-1)^{1-2\gamma} \int_s^t \mathcal{D}^{1-2\gamma}_{s+} \delta Z'(u) \mathcal{R}^{(2,\gamma)}_{t-} X(u) \, du \end{aligned} \tag{2}$$

for $(s,t) \in \Delta_T$. Here $\delta Z'_{s,t} := Z'_t - Z'_s$, $\mathcal{R}^{(1,\gamma)}_{t-} X(u) := \mathcal{D}^\gamma_{t-} X^1(u)$ and

$$\mathcal{R}^{(2,\gamma)}_{t-} X(u) := \mathcal{D}^{2\gamma}_{t-} X^2(u) + \frac{(-1)^\gamma \gamma}{\Gamma(1-\gamma)} \int_u^t \frac{X^1_{u,v} \otimes \mathcal{R}^{(1,\gamma)}_{t-} X(v)}{(v-u)^{\gamma+1}} \, dv.$$

We refer to [2] for the details of $I^\gamma(X, Z)$ and the generalization for any $\beta \in (0, 1]$.

## References


[1] M. Gubinelli, Controlling rough paths, *J. Funct. Anal.* **216** (2004), 86–140.

[2] Y. Ito, Extension theorem for rough paths via fractional calculus, preprint.



Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan

*E-mail address*: itoyu@acs.i.kyoto-u.ac.jp