A central limit theorem for non-symmetric random walks on crystal lattices

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This talk is based on a joint work with Hiroshi Kawabi(Okayama Univ.) and Motoko Kotani (Tohoku Univ.).

A locally finite, connected oriented graph $X = (V, E)$ is called *crystal lattice* if X is an abelian covering graph of a finite graph $X_0 = (V_0, E_0)$. We denote by $\Gamma \simeq \mathbb{Z}^d$ the covering transformation group. Our interest is the long time behavior of the transition probability

$$p(n, x, y) = \sum_{\substack{(e_1, e_2, \dots, e_n) \in C_{x,n} \\ t(e_n) = y}} p(e_1) p(e_2) \cdots p(e_n)$$

given by a 1-step transition probability $p : E \to [0, 1]$ satisfying

$$\sum_{e \in E_x} p(e) = 1, \quad p(e) + p(\overline{e}) > 0, \quad \forall \sigma \in \Gamma,\, p(\sigma e) = p(e).$$

There are many results of this problem under some various settings. See Spitzer [10], Lawler [9] and references therein. Our study is motivated by the following local central limit theorem (LCLT) presented by Sunada [11]:

Theorem 0.1 *Suppose that the random walk is irreducible with period K. Then*

$$p(n, x, y) \sim \frac{K \mathrm{vol}(\mathrm{Alb}^\Gamma) m(y)}{(2\pi n)^{d/2}} \exp\left(-\frac{\|\Phi(y) - \Phi(x) - n\rho_{\mathbb{R}}(\gamma_p)\|^2}{2n} \right),$$

where m is the (lift of) normalized invariant measure on X_0, γ_p is the homological direction, $\rho_{\mathbb{R}}$ is the canonical surjective homomorphism from $\mathrm{H}_1(X_0, \mathbb{R})$ to $\Gamma \otimes \mathbb{R}$, $\Phi : X \to \Gamma \otimes \mathbb{R}$ is the modified harmonic realization, defined by

$$\forall x \in V, \quad \Delta \Phi(x) := \sum_{e \in E_x} p(e) \left(\Phi(o(x)) - \Phi(t(e)) \right) = \rho_{\mathbb{R}}(\gamma_p),$$

and $\| \cdot \|$ is the Albanese metric on $\Gamma \otimes \mathbb{R}$, induced by

$$\begin{array}{ccc} \Gamma \otimes \mathbb{R} & \longleftarrow & \mathrm{H}_1(X_0, \mathbb{R}) \\ \updownarrow & & \updownarrow \\ \mathrm{Hom}(\Gamma, \mathbb{R}) & \hookrightarrow & \mathrm{H}^1(X_0, \mathrm{R}) \simeq \mathcal{H}^1(X_0). \end{array}$$

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Here $\mathcal{H}^1(X_0)$ is the space of modified harmonic 1-forms difined by

$$\forall x \in V_0, \quad \delta\omega(x) + \langle \gamma_p, \omega \rangle = 0$$

equipped with a canonical inner product defined by

$$\langle\langle \omega_1, \omega_2 \rangle\rangle = \sum_{e \in E_0} p(e)\omega_1(e)\omega_2(e)m(o(e)) - \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle.$$

See also .[2], [3], [4], [5], [6], [7], [8], [12].

It is natural to ask the weak convergence of the sequence of law of the probability measure of the random walk on X. In this talk we give two canonical weak convergences.

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